PROCEEDINGS

OF THE

EDINBURGH MATHEMATICAL SOCIETY.

VOLUME I.

SESSION 1883.

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INDEX.

ERRATA.

PAGE 7. Last line but one, W. J. should be J. W.

„ 18. Sixth line, MM should be MM′.

„ 23. Last line but one, maximum should be minimum.

„ 48. The following historical note in reference to (5) has been omitted:—

As regards the expressions for s, s_1, s_2, s_3 some are given in Heron's theorem regarding the area of a triangle; some are given by Euler in *Novi Commentarii Academiae...Petropolitanae* for the years 1747–8, I. 49–66 (1750). All except those where D, E, F occur are given in the *Ladies' Diary* for 1759; all are given by T. S. Davies in the *Ladies' Diary* for 1835, pp. 51–2.

Six of the expressions for a, b, c are given by T. S. Davies in the *Ladies' Diary* for 1835, p. 52.

PROCEEDINGS

OF THE

EDINBURGH MATHEMATICAL SOCIETY.

FIRST SESSION, 1883.

The EDINBURGH MATHEMATICAL SOCIETY owes its origin to the suggestion of the late Mr A. Y. FRASER, at that time one of the mathematical masters in George Watson's College, Edinburgh. With him were closely associated Mr A. J. G. BARCLAY, now of Glasgow High School, and Dr C. G. KNOTT of the University of Edinburgh. The following is the circular which they issued to those persons throughout the kingdom whom they deemed likely to take an interest in such a Society:

EDINBURGH UNIVERSITY,
January 23, 1883.

DEAR SIR,

We, the undersigned, beg to call your attention to the following proposal, in the hope that you will find it in your power to give it your support:

It is proposed to establish, primarily in connection with the University, a Society for the mutual improvement of its members in the Mathematical Sciences, pure and applied.

Amongst the methods by which this object might be attained may be mentioned: Reviews of works both British and Foreign, historical notes, discussion of new problems or new solutions, and comparison of the various systems of teaching in different countries, or any other means tending to the promotion of Mathematical Education.

It is suggested that the Society be formed, in the first instance, of all those who shall give in their names on or before February 2, 1883, and who are

(1) present or former students in either of the Advanced Mathematical Classes of Edinburgh University,

(2) Honours Graduates in any of the British Universities, or

(3) recognised Teachers of Mathematics;

and that, after the above mentioned date, members be nominated and elected by ballot in the usual manner.

It may be added that Professors Tait and Chrystal have expressed themselves as highly favourable to the project, as one that may lead to important results.

If there are any of your friends who might take an interest in the Society, kindly inform them of its objects, and invite them to attend the Preliminary Meeting, to be held in the MATHEMATICAL CLASS ROOM here, on Friday, February 2, 1883, at Eight P.M., at which meeting your presence is respectfully requested.

We are,

Yours faithfully,

CARGILL G. KNOTT, D.Sc. (Edin.), F.R.S.E.
A. J. G. BARCLAY, M.A. (Edin.).
A. Y. FRASER, M.A. (Aberdeen).

Preliminary Meeting, February 2nd, 1883.

C. G. KNOTT, Esq., D.Sc., F.R.S.E., in the Chair.

At this meeting it was agreed to institute a Society to be called "The Edinburgh Mathematical Society."

The following Office-Bearers were elected:—

President—Mr JOHN S. MACKAY, M.A., F.R.S.E.

Vice-President—Mr ALEXANDER MACFARLANE, M.A., D.Sc., F.R.S.E.

Secretary and Treasurer—Mr CARGILL G. KNOTT, D.Sc., F.R.S.E.

Committee.

Messrs R. E. ALLARDICE, M.A.; A. J. G. BARCLAY, M.A.; A. Y. FRASER, M.A.; W. J. MACDONALD, M.A.

The Committee were instructed to draw up a constitution for the Society, and to submit the same at the first ordinary meeting.

First Meeting, March 12th, 1883.

JOHN S. MACKAY, Esq., M.A., F.R.S.E., President, in the Chair.

Professor CHRYSTAL, University of Edinburgh, gave an introductory address on "Present Fields of Mathematical Research."

Thereafter, the Society proceeded to the discussion of the draft constitution submitted by the Committee.

Second Meeting April 13th, 1883.

JOHN STURGEON MACKAY, Esq., M.A., F.R.S.E., President, in the Chair.

The Triangle and its Six Scribed Circles.

By JOHN STURGEON MACKAY, M.A.

[When the Edinburgh Mathematical Society was founded, it was deemed impracticable, from the expense that would be involved, to print the papers read at its meetings. It was, however, resolved that copies of them should be deposited with the Secretary, and that these copies should as far as possible be made accessible to the members of the Society. Want of leisure during the first session prevented me from doing my part in carrying out this intention of the Society, and committing to writing the paper which was delivered at the second meeting. During its second session, the Society resolved to print its proceedings in whole or in abstract; and a beginning was made with the second volume, the first being left over for future consideration, as the cost of printing absorbed nearly the entire income of the Society.

Some time afterwards, through the exertions of the late Mr A. Y. Fraser, who was then Secretary of the Society, and whose opening career was cut short by untimely death, a sum of money was raised by voluntary subscription to defray the expense of printing Vol. I. Seeing that it was desired by others as well as by myself to have the collection of properties of the triangle as ample as possible, I obtained the consent of the Society to delay the publication of the volume. This delay has been unduly prolonged, and no one but myself is to be blamed in connection with it.

The collection, divided into twenty sections, which has resulted is a tolerably large one, filling somewhat more than 1600 quarto pages of manuscript. Owing to the impossibility of printing all this without a large expenditure of money, that portion has been selected which best corresponds with what was communicated to the Society. The six scribed circles of the title were the inscribed, the escribed,

the circumscribed, and, as I called it then, the medioscribed (or nine point) circles. Part of the section treating of the nine point circle was printed in abstract in Vol. II. of the *Proceedings*, and the rest, considerably enlarged, in Vol. XI. What is here printed consists of the first section, with the principal properties of some other sections incorporated in it, and a statement of the notation which is employed throughout all the sections. The remainder of the collection will, unless it is anticipated somewhere else, see the light gradually, I hope, in the *Proceedings* of the Society.

This fragmentary memoir would have appeared more original if I had left without note or comment all those properties which I had discovered for myself, and it would certainly have looked more learned if I had furnished references to all the authorities who have discussed various parts of the subject. Unless there was reason to the contrary, as, for example, when a new mode of proof was given, I have noted simply the first discoverer of a theorem, and have suppressed all mention of subsequent discoverers. I have not concerned myself with problems, with questions of loci, or with trigonometrical expressions relating to the triangle, for the simple reason that I wished to keep the memoir within manageable bounds. The same reason has also led me to defer consideration of the properties of the triangle, in which other conic sections than the circle are involved.

The research which has been necessary to render the historical notes scattered throughout the whole collection worthy of attention could not, in the limited leisure at my disposal, have been undertaken except with the aid of many friends. I gladly seize the present opportunity of recording my gratitude for many favours to my colleagues of the Edinburgh Mathematical Society, Messrs Alison, Allardice, Gibson, Harvey, Macdonald, Pressland, and, in particular, the late Mr A. Y. Fraser; to Messrs Anderson, Langley, and Tucker in England; to Professor Neuberg in Belgium; to Dr Grebe and Professor Fuhrmann in Germany: and to Messrs Aubert, Lemoine, De Longchamps, D'Ocagne, and Poulain in France. For two successive summers Messrs Nony and Vuibert put a complete copy of their *Journal de Mathématiques Élémentaires* in my hands, with liberty to make such extracts from it as I pleased; and Mr De Longchamps, with the courtesy of his nation, has several times placed his whole library at my disposal, even when he was absent from Paris.]

NOTATION

A few preliminary words are necessary in respect to the following scheme of notation.

In the first place the notation should be uniform, that is to say, when a definite point connected with the triangle is under consideration it should always be denoted by the same letter, and not by one letter at one time and by another letter at another. But the converse practice of never using a letter which has denoted one special point to denote any other point need not be carried out, and indeed cannot be, unless the number of properties investigated be very small, or unless recourse be had to letters ornamented with an intolerable number of suffixes or accents. To illustrate by an example. When the circumcentre of a triangle ABC has to be mentioned, it is invariably called O, but in cases where the circumcentre does not come into consideration at all, there is no adequate reason why O should not be used to denote some other point such as the point of concurrency of three angular transversals.

In the second place capital letters, loaded or not as the case may be with an accent or a suffix or even with both, should denote points, and small letters should denote lines. I am aware that such a rule is not adhered to in some of the best geometrical treatises, for instance those of Chasles who constantly uses both capitals and small letters to denote points. The extremely prevalent, though not universal, practice of denoting the radius of the circumcircle by R shows that one deviation at least must be made from the rule if accordance with general usage is to be secured. For the distances between the incentre, the excentres, and the vertices of a triangle Greek letters have been employed. This was found to be unavoidable, if conciseness was to be aimed at. These Greek letters are the notation adopted by the discoverers of not a few of the properties* connected with the incentre, etc., and I could not invent a better.

In the third place the notation should conform as far as may be to that actually in use among geometers. I have had this consideration continually before me, and it has caused me a world of trouble. For the notation has been changed, rechanged, and changed again; and though I dare not hope that the compromise which has been

* The properties referred to are not contained in the section which is here printed.

come to will commend itself to everybody as the best in the circumstances, yet if it were needful I could support with a considerable weight of authority drawn from one country or another, the designation of every important point to which objection might be taken.

POINTS.

A, B, C,	= vertices of the fundamental triangle. When the sides of triangle ABC are spoken of, they are understood to be taken in the order BC, CA, AB.
A′, B′, C′	= mid points of the sides BC, CA, AB.
A_∞, B_∞, C_∞	= harmonic conjugates of A′, B′, C′, for* BC, CA, AB.
A_1, B_1, C_1 A_2, B_2, C_2	= vertices of some triangle related to ABC, for example, anticomplementary, inscribed or circumscribed.
D, E, F	= points of contact of sides with incircle. = points of intersection of sides with transversal. = points of intersection of sides with angular transversals. = projections of a point on the sides.
D_1, E_1, F_1 D_2, E_2, F_2 D_3, E_3, F_3	= points of contact of sides with 1st excircle. 2nd excircle. 3rd excircle.
D′, E′, F′	= harmonic conjugates of D, E, F for BC, CA, AB. = those points in A′B′C′ which correspond to D, E, F in ABC. Similarly for D_1', E_1', F_1', etc.
G, G_a, G_b, G_c	= centroids of ABC, HCB, CHA, BAH.
G_0, G_1, G_2, G_3	= centroids of $I_1I_2I_3$, II_3I_2, I_3II_1, I_2I_1I.

* Following the example of Mr W. J. Russell, I have used the word "for" instead of the phrase "with respect to."

Γ	= point of concurrency of AD, BE, CF.
$\Gamma_1, \Gamma_2, \Gamma_3$	= points of concurrency of AD_1, BE_1, CF_1, and so on.
	These four points are frequently called the Gergonne points of ABC.
H	= orthocentre of ABC.
I	= incentre of ABC.
I_1, I_2, I_3	= 1st, 2nd, 3rd excentres of ABC.
J	= incentre of $A_1B_1C_1$, the triangle anticomplementary to ABC.
J_1, J_2, J_3	= 1st, 2nd, 3rd excentres of $A_1B_1C_1$.
	These four points are frequently called the Nagel points of ABC.
K	= insymmedian point of ABC (Lemoine's point).
K_1, K_2, K_3	= 1st, 2nd, 3rd exsymmedian points of ABC.
L, M, N	= feet of internal angular bisectors of ABC.
	= projections of the symmedian point on the sides of ABC.
L′, M′, N′	= feet of external angular bisectors of ABC.
L_1, M_1, N_1	= projections of the first exsymmedian point on the sides of ABC.
	Similarly for L_2, M_2, N_2, etc.
O	= circumcentre of ABC.
	= point of intersection of three concurrent lines.
O′	= circumcentre of A′B′C′, or nine-point centre of ABC.
O_a, O_b, O_c	= circumcentres of HCB, CHA, BAH.
O_0, O_1, O_2, O_3	= circumcentres of $I_1I_2I_3$, II_3I_2, I_3II_1, I_2I_1I.
O_1, O_2, O_3	= circumcentres of ΩCA, ΩAB, ΩBC.
O_1', O_2', O_3'	= circumcentres of Ω'CA, Ω'AB, Ω'BC.
O, O′	= pairs of isogonal points with respect to ABC.

P, Q P, P′ Q, Q′	= pairs of special points.
R, S, T	= points where the perpendiculars of a triangle meet the circumcircle. = feet of the insymmedians. = projections of Ω on BC, CA, AB.
R′, S′, T′	= feet of the exsymmedians. = projections of Ω' on BC, CA, AB.
T, T_1, T_2, T_3	= points of contact of the nine-point circle with the incircle and the excircles. = centres of the four Taylor circles.
U, U′	= ends of that diameter of the circumcircle which is perpendicular to BC. U is on the opposite side of BC from A. Similarly for V, V′ and W, W′.
U, V, W	= points in which concurrent lines from A, B, C meet the circumcircle ABC. = mid points of AH, BH, CH.
X, Y, Z	= feet of the perpendiculars from A, B, C.
X′, Y′, Z′	= harmonic conjugates of X, Y, Z for BC, CA, AB. = mid points of YZ, ZX, XY.
Ω, Ω'	= Brocard points of ABC.

Regarding the notation which may be adopted for many other important points connected with the triangle, such as the isogonic and isodynamic centres, the centres of Brocard's circle and Lemoine's first circle (triplicate ratio), Steiner's point, Tarry's point, and so on, no suggestion is offered for the present. Those who wish to see the notations which have been employed may be referred to Mr De Longchamps' *Journal de Mathématiques Élémentaires* and to *Mathesis*.

LINES.

$a,\ b,\ c$	= the sides BC, CA, AB of ABC.
$\alpha,\ \beta,\ \gamma$	= AI, BI, CI.
$\alpha_1,\ \beta_1,\ \gamma_1$	= AI_1, BI_1, CI_1.
$\alpha_2,\ \beta_2,\ \gamma_2$	= AI_2, BI_2, CI_2.
$\alpha_3,\ \beta_3,\ \gamma_3$	= AI_3, BI_3, CI_3.
$\alpha_1-\alpha,\ \beta_2-\beta,\ \gamma_3-\gamma$	= II_1, II_2, II_3.
$\alpha_2+\alpha_3,\ \beta_3+\beta_1,\ \gamma_1+\gamma_2$	= I_2I_3, I_3I_1, I_1I_2.
$d,\ e,\ f$	= the sides EF, FD, DE of DEF. Similarly for the sides of $D_1E_1F_1$, etc.
$h_1,\ h_2,\ h_3$	= the perpendiculars AX, BY, CZ.
$h_1',\ h_2',\ h_3'$	= the segments AH, BH, CH of the perpendiculars.
$h_1'',\ h_2'',\ h_3''$	= the segments HX, HY, HZ of the perpendiculars.
$k_1,\ k_2,\ k_3$	= the perpendiculars OA′, OB′, OC′ from the circumcentre.
$l_1,\ l_2,\ l_3$	= the internal angular bisectors of A, B, C.
$\lambda_1,\ \lambda_2,\ \lambda_3$	= the external angular bisectors of A, B, C.
$m_1,\ m_2,\ m_3$	= the medians from A, B, C.
$n_1,\ n_2,\ n_3$	= the insymmedians from A, B, C.
$\nu_1,\ \nu_2,\ \nu_3$	= the exsymmedians from A, B, C.
$p_1,\ p_2,\ p_3$	= perpendiculars from any point on BC, CA, AB.
r	= radius of the incircle.
$r_1,\ r_2,\ r_3$	= radii of the 1st, 2nd, 3rd excircles.

R	= radius of the circumcircle.
R_1, R_2, R_3	= radii of the circumcircles of Ω CA, Ω AB, Ω BC.
R_1', R_2', R_3'	= radii of the circumcircles of Ω'CA, Ω'AB, Ω'BC.
ρ	= radius of the incircle of XYZ,
ρ_1, ρ_2, ρ_3	= radii of the 1st, 2nd, 3rd excircles of XYZ.
s	= semiperimeter of ABC.
s_1, s_2, s_3	$= s - a$, $s - b$, $s - c$.
x, y, z	= the sides YZ, ZX, XY of XYZ.

AREAS.

Δ, Δ_a, Δ_b, Δ_c	= ABC, HCB, CHA, BAH.
Δ_0, Δ_1, Δ_2, Δ_3	$= I_1I_2I_3$, $I\,I_3I_2$, $I_3\,I\,I_1$, $I_2\,I_1\,I$.

SECTION I.

§ 1. Centroid.

§ 2. Circumcentre.

§ 3. Incentre.

§ 4. Excentres.

§ 5. Orthocentre.

§ 6. Euler's Line.

§ 7. Relations among Radii.

§ 8. Area.

§ 1. Centroid.

*The medians of a triangle are concurrent.**

Figure 1.

Let the medians BB′, CC′ cut each other at G; join AG, and let it cut BC at A′.

Produce AA′ to L, making GL equal to GA, and join BL, CL.

Because	C′G bisects AB and AL,
therefore	C′G is parallel to BL.
Similarly	B′G „ „ „ CL;
therefore	BLCG is a parallelogram;
therefore	A′ is the mid point of BC.

This theorem may be proved in many other ways.

Def.—The point G is called sometimes the *centre of gravity* † of the triangle ABC; sometimes the *centre of mean distances* ‡ of the points A, B, C; and more frequently now the *centroid* § of the triangle ABC.

The simplest construction for obtaining G by means of the ruler and the compasses is the following ||:—

With B as centre and AC as radius describe a circle; with C as centre and AB as radius describe a second circle cutting the former below the base at D. Join DC and produce it to meet the second circle at E.

AD and BE intersect at the centroid G.

(1) $A'G = \frac{1}{2}AG = \frac{1}{3}AA'$.

Hence the centroid of a triangle may be found by drawing any median and trisecting it; and if two (or a series of) triangles have the same vertex and the same median drawn from that vertex, they have the same centroid.

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* Archimedes, *De planorum æquilibriis*, I. 13, 14.

† Archimedes.

‡ Carnot, *Géométrie de Position*, p. 315 (1803), and Lhuilier, *Élémens d'Analyse*, p. X. (1809).

§ This expression was suggested by T. S. Davies in 1843 in the *Mathematician* I. 58. It had been used by Dr Hey in 1814 to designate another point.

|| Mr E. Lemoine in the Report (second part) of the 21st session of the *Association Française pour l'avancement des sciences*, p. 77 (1892).

(2) Triangle GBC = GCA = GAB = $\frac{1}{3}$ABC.

(3) The sides of triangle A'B'C' are respectively parallel to those of ABC; hence these triangles are directly similar.

Also, since the lines AA', BB', CC' joining corresponding vertices are concurrent at G, triangles ABC, A'B'C' are homothetic, and G is their homothetic centre.

Def.—Triangles such as the fundamental triangle ABC, and that formed by joining the feet of its medians have in recent years received the following names:—

A'B'C' is the *complementary* triangle of ABC.
ABC „ „ *anticomplementary* „ „ A'B'C'.

These names are applied also to corresponding points * in such triangles. Thus if P be any point in or outside of triangle ABC, and P' be the corresponding point in or outside of triangle A'B'C',

P' is the *complementary* point of P,
P „ „ *anticomplementary* „ „ P'.

(4) If $A_1B_1C_1$ be the triangle formed by drawing through A, B, C parallels to the opposite sides of triangle ABC,

ABC is the complementary triangle of $A_1B_1C_1$,
$A_1B_1C_1$ „ „ anticomplementary „ „ ABC.

Figure 2.

(5) The fundamental triangle ABC is directly similar to the triangles cut off from it by the sides of its complementary triangle, AC'B', C'BA', B'A'C.

(6) The centroid of the fundamental triangle is the centroid of the complementary triangle; the centroid of the complementary triangle is the centroid of its complementary triangle; and so on.

(7) All straight lines parallel to the base of a triangle and terminated by the other sides are bisected by the median to the base.

* See Mr Emile Vigarié's articles *Sur les Points Complémentaires* in *Mathesis*, VII., 5-12, 57-62, 84-9, 105-110 (1887).

Hence, if EF, GH, KL ... be parallel to BC, the points E, G, K ... being on AC, and F, H, L ... on AB, the intersections of

BE, CF; BG, CH; BK, CL; FG, EH; FK, EL ...... will all lie on the median * from A.

(8) If two triangles have the same base, the straight line which joins their vertices is parallel to and three times as long as the straight line which joins their centroids.

(9) If G be any point in the plane of ABC, and G_a, G_b, G_c be the centroids of triangles GBC, GCA, GAB, triangle $G_aG_bG_c$ is directly similar † to triangle ABC.

(10) If P be any point on the circumcircle of ABC, the centroids of the four triangles PBC, PCA, PAB, ABC are concyclic. ‡

For if the centroids of these triangles be denoted by D, E, F, G respectively, the quadrilateral DEFG has its sides DE, EF, FG, GD respectively parallel to BA, CB, PC, AP, and one-third as long.

Mr Griffiths states § that if the circle on which the four centroids lie be called the centroid-circle of the quadrangle ABCP, it may be shown that the centroid-circles of the five quadrangles that can be formed from five concyclic points will also have their centres on the circumference of another circle of one-third the radius of the first.

Townsend gives the following generalisation || of (10):

If A, B, C, D, E, F, etc., be the position of any number (n) of equal masses distributed in space, G that of their centre of gravity, and A′, B′, C′, D′, E′, F′, etc., those of the centres of gravity of their n different groups of $(n-1)$; then always the two systems of n points A, B, C, D, E, F, etc., and A′, B′, C′, D′, E′, F′, etc., are similar, oppositely placed with respect to each other, have G for their centre of similitude, and $(n-1):1$ for their ratio of similitude.

The truth of this is evident, for the several lines AA′, BB′, CC′, DD′, EE′, FF′, etc., all connect through G, and are then divided internally in the common ratio of $(n-1):1$.

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* Jacobi, *De Triangulorum rectilineorum proprietatibus*, pp. 5-6 (1825).

† Professor R. E. Allardice.

‡ Mr J. Griffiths, in *Mathematical Questions from the Educational Times*, V. 92 (1866).

§ *Notes on the Geometry of the Plane Triangle*, p. 65 (1867).

|| *Mathematical Questions from the Educational Times*, V. 92 (1866).

DEF.—If the vertex A of a triangle ABC be joined to any point D in the base, the fourth harmonic ray to AB, AD, AC is found by dividing BC externally at D′ in the ratio BD : CD, and joining AD′.

When the point D is the mid point of BC, namely A′, the fourth harmonic ray to AB, AA′, AC is the line through A parallel to BC, and it may be denoted by AA_∞.

Similarly, the line through B parallel to CA will be the fourth harmonic ray to BC, BB′, BA, and may be denoted by BB_∞; the line through C parallel to AB will be the fourth harmonic ray to CA, CC′, CB, and may be denoted by CC_∞.

If therefore AA′, BB′, CC′ be called the *internal medians* of triangle ABC, then AA_∞, BB_∞, CC_∞ may be called the *external medians.*

(11) The six medians, internal and external, of a triangle meet three and three in four points, which are the centroid and the points anticomplementary to the vertices of the triangle, namely G, A_1, B_1, C_1.

FIGURE 2.

DEF.—The points A_1, B_1, C_1, G form a *tetrastigm* (a system of four points, no three of which are collinear), and the three pairs of opposite connectors,

$$A_1G,\ B_1C_1;\quad B_1G,\ C_1A_1;\quad C_1G,\ A_1B_1$$

meet in A ; B ; C,

which are the *centres* of the tetrastigm, and ABC is the *central triangle* of the tetrastigm.

If ABCG be the tetrastigm, the points A′, B′, C′ are its centres, and A′B′C′ its central triangle.

(12) *If in the internal median AA′ of triangle ABC any point M be taken, and MP, MQ be drawn perpendicular to AC, AB, then MP, MQ are inversely proportional to AC, AB.*

FIGURE 3.

Join MB, MC.

Then $AMB = AMC$;

therefore $AB \cdot MQ = AC \cdot MP$;

therefore $AB : AC = MP : MQ$.

(13) *If in the external median AA_∞ of triangle ABC any point M′ be taken, and M′P′, M′Q′ be drawn perpendicular to AC, AB, then M′P′, M′Q′ are inversely proportional to AC, AB.*

FIGURE 4.

Join M′B, M′C.

$$\begin{array}{ll} \text{Then} & AM'B = AM'C; \\ \text{therefore} & AB \cdot M'Q' = AC \cdot M'P'; \\ \text{therefore} & AB : AC = M'P' : M'Q'. \end{array}$$

(14) If from G the centroid of ABC there be drawn p_1, p_2, p_3 perpendicular to BC, CA, AB, then

$$BC : CA : AB = \frac{1}{p_1} : \frac{1}{p_2} : \frac{1}{p_3}.$$

(15) If from G the centroid of ABC there be drawn p_1', p_2', p_3' perpendicular to B_1C_1, C_1A_1, A_1B_1, then

$$BC : CA : AB = \frac{1}{p_1'} : \frac{1}{p_2'} : \frac{1}{p_3'}.$$

(16) *If the vertex A of the triangle ABC falls on the base BC, the centroid G of the three collinear points A, B, C is found by the construction indicated in* (1):

Bisect BC in A′, and divide AA′ at G so that

$$AG : A'G = 2 : 1.$$

In this case the sum of the distances from the centroid of the points on one side of it is equal to the distance from the centroid of the point on the opposite side.

FIGURE 5.

For

$$\begin{aligned} AG + BG &= (AA' - GA') + (BA' - GA'), \\ &= AA' + BA' - 2GA', \\ &= AA' + CA' - GA, \\ &= CA - GA, \\ &= CG. \end{aligned}$$

(17) If ABC be a triangle, G its centroid, and A′, B′, C′, G′ the projections of A, B, C, G, on any straight line XY, then G′ is the centroid of the three collinear points A′, B′, C′.

(18) LEMMA.*

If a straight line BC be divided internally at M so that

$$BM : CM = n : m$$

and if from B, M, C perpendiculars BB′, MM′, CC′, be drawn to any straight line XY, then

$$(m + n)MM = mBB' \pm nCC'$$

the upper sign being taken when B and C are on the same † *side of XY, and the lower when they are on opposite sides of XY.*

FIGURE 6.

Join BC′ meeting MM′ in N.

The triangles BB′C, NM′C are similar;

therefore $\mathrm{BB'} : \mathrm{NM'} = \mathrm{BC'} : \mathrm{NC'}$

$= \mathrm{BC} : \mathrm{MC}$

$= m + n : m$;

therefore $m\mathrm{BB'} = (m + n)\mathrm{NM'}$.

The triangles BCC′, BMN are similar;

therefore $\mathrm{CC'} : \mathrm{MN} = \mathrm{BC} : \mathrm{BM}$,

$= m + n : n$;

therefore $n\mathrm{CC'} = (m + n)\mathrm{MN}$.

Hence $m\mathrm{BB'} + n\mathrm{CC'} = (m + n)\mathrm{NM'} + (m + n)\mathrm{MN}$

$= (m + n)\mathrm{MM'}$.

(19) *The distance of the centroid of a triangle from any straight line is an arithmetic mean between the distances of the vertices from the same straight line.*

FIGURE 7.

Let AM be the median from A, and G the centroid.

Take A′, B′, C′, G′, M′ the projections of A, B, C, G, M on any straight line XY.

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* Lhuilier, *Élémens d'Analyse*, pp. 1-2 (1809).

† The figure and demonstration refer only to this case. The other case and the consideration of what happens when BC is divided externally are left to the reader.

Because	$BM : CM = 1 : 1,$
therefore	$BB' + CC' = 2MM'.$
Because	$AG : MG = 2 : 1,$
therefore	$AA' + 2MM' = 3GG';$
therefore	$AA' + BB' + CC' = 3GG'.$

The figure and demonstration refer only to the case when A, B, C are all on the same side of XY. If A and B be on the same side of XY, and C on the opposite side, the result will be

$$AA' + BB' - CC' = 3GG'.$$

When XY passes through the centroid G, the sum of the distances from XY of the vertices on one side of it is equal to the distance from XY of the vertex on the opposite side.

For a very full account of the properties of the centre of mean distances see the preliminary dissertation in Lhuilier's *Élémens d'Analyse* (1809), and Townsend's *Modern Geometry of the Point, Line, and Circle*, I. 117-143 (1863).

(20) *The sum of the squares of the distances of the vertices of a triangle from any point is equal to the sum of the squares of their distances from the centroid increased by three times the square of the distance between the point and the centroid.**

FIGURE 8.

Let G be the centroid of ABC, and P any other point. Join PG, and on it draw perpendiculars from A, B, C.

Then
$$AP^2 = AG^2 + PG^2 - 2PG\cdot A'G,$$
$$BP^2 = BG^2 + PG^2 + 2PG\cdot B'G,$$
$$CP^2 = CG^2 + PG^2 - 2PG\cdot C'G;$$
therefore
$$AP^2 + BP^2 + CP^2 = AG^2 + BG^2 + CG^2 + 3PG^2 - 2PG(A'G - B'G - C'G),$$
$$= AG^2 + BG^2 + CG^2 + 3PG^2,$$
since
$$A'G - B'G - C'G = 0.$$

(21) If a circle be described with G as centre, and any radius, and any two points P, Q be taken on its circumference †

$$AP^2 + BP^2 + CP^2 = AQ^2 + BQ^2 + CQ^2.$$

---

* This is a particular case of a more general theorem proved in Robert Simson's *Apollonii Pergaei Locorum Planorum Libri II.*, pp. 179-180 (1749).

† C. F. A. Jacobi, *De Triangulorum Rectilineorum Proprietatibus*, p. 7 (1825).

(22) *That point the sum of the squares of whose distances from the vertices of a triangle is a minimum is the centroid of the triangle.**

If the triangle ABC is fixed, G is a fixed point, and AG, BG, CG fixed distances. Hence for any variable point P, $AP^2 + BP^2 + CP^2$ always exceeds the constant quantity $AG^2 + BG^2 + CG^2$ by $3PG^2$. The nearer therefore P approaches to G, the nearer does $AP^2 + BP^2 + CP^2$ approach this constant quantity.

(23) *That point inside a triangle which has the product of its distances from the three sides a maximum is the centroid of the triangle.*†

Let G be any point inside ABC, and GR, GS, GT its distances from BC, CA, AB.

Then $GR \times GS \times GT$ is a maximum,
when $GR\cdot\frac{1}{2}BC \times GS\cdot\frac{1}{2}CA \times GT\cdot\frac{1}{2}AB$ is a maximum,
that is, when $GBC \times GCA \times GAB$ is a maximum,
that is, when these three triangles are equal,
that is, when G is the centroid.

(24) *If three straight lines drawn from the vertices of a triangle are concurrent, the three straight lines drawn parallel to them from the mid points of the opposite sides are also concurrent; and the straight line joining the two points of concurrency passes through the centroid of the triangle and is there trisected.* ‡

The triangles ABC, A′B′C′ are similar and oppositely situated, G is their homothetic centre, and 2 : 1 is the ratio of similitude.

Hence if AD, BE, CF be concurrent at O, the corresponding straight lines A′D′, B′E′, C′F′ will pass through the corresponding point O′; OO′ the straight line joining two corresponding points, will pass through the homothetic centre G; and OG : O′G = 2 : 1.

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* J. F. de Tuschis a Fagnano in *Nova Acta Eruditorum, anni* 1775, p. 290. The article referred to is entitled: *Problemata quaedam ad methodum maximorum et minimorum spectantia,* and the volume in which it occurs was published at Leipzig in 1779.

† H. Watson in the *Ladies' Diary* for 1756.

‡ Frégier in Gergonne's *Annales*, VII. 170 (1816-7).

(25) *If ABC be a triangle, O any point whatever, and A_1, B_1, C_1 symmetrical to O with respect to the mid points of BC, CA, AB, then* *

(a) *AA_1, BB_1, CC_1 are concurrent at a point P.*

(b) *The straight line OP turns round a fixed point G when the point O moves in any manner whatever.*

(c) *The point G divides OP in a constant ratio.*

FIGURE 9.

Let A′, B′, C′ be the mid points of BC, CA, AB.

(*a*) Then A_1B_1 is parallel to A′B′ and equal to 2A′B′;
therefore it is equal and parallel to AB, but oppositely directed.
Similarly B_1C_1 and C_1A_1 are equal and parallel to BC and CA, but oppositely directed.
The three pairs of parallels BC and B_1C_1, CA and C_1A_1, AB and A_1B_1 form therefore three parallelograms, whose diagonals AA_1, BB_1, CC_1 cut each other at P the mid point of each of them.

(*b*) In triangle OAA_1 the lines OP, AA′ are medians;
therefore OP cuts AA′ at G such that AG = 2A′G.
But AA′ is a median of triangle ABC;
therefore G is the centroid of ABC, and consequently a fixed point.

(*c*) OP is divided at G so that OG = 2GP.

(26) The sum of the squares on the sides of the complementary triangle is one-fourth of the sum of the squares on the sides of the fundamental triangle.

(27) If in a triangle its complementary triangle be inscribed, and in the complementary triangle its complementary triangle be inscribed, and so on, the limit of the sum of the squares on the sides of all the triangles so formed is one-third of the sum of the squares on the sides of the fundamental triangle.†

(28) If in a triangle its complementary triangle be inscribed, and so on, the limit to which these triangles tend is a point, and the sum of the squares on the lines drawn therefrom to the vertices of

* Mr Maurice d'Ocagne in the *Nouvelles Annales*, 3rd Series, I. 239 (1882); proof on p. 430.

† Leybourn's *Mathematical Repository*, new series, V. 111 (1820).

all the inscribed triangles is one-third of the sum of the squares on the lines drawn from the same point to the vertices of the fundamental triangle.*

(29) *If $A_1B_1C_1$ be the complementary triangle of ABC, $A_2B_2C_2$ the complementary triangle of $A_1B_1C_1$, and so on; and if P be any point in the plane of the triangle, then*†

$$PA_n^2 + PB_n^2 + PC_n^2 = 3PG^2 + \frac{1}{3\cdot 4^n}(BC^2 + CA^2 + AB^2).$$

FIGURE 10.

Join P with A, B, C, A′, G, and with D the mid point of AG.

$$\begin{aligned} \text{Then}\qquad AB^2 + AC^2 &= 2A_1A^2 + 2A_1B^2; \\ &= 18A_1G^2 + 2A_1B^2; \end{aligned}$$

$$\text{therefore}\quad BC^2 + CA^2 + AB^2 = 18A_1G^2 + 6A_1B^2.$$

$$\begin{aligned} \text{Again,}\qquad PG^2 + PA^2 &= 2\,(GD^2 + PD^2), \\ PB^2 + PC^2 &= 2\,(A_1B^2 + PA_1^2), \\ 2\,(PA_1^2 + PD^2) &= 4\,(A_1G^2 + PG^2); \end{aligned}$$

therefore by addition, and subtraction of what is common,

$$\begin{aligned} PA^2 + PB^2 + PC^2 &= 3PG^2 + 6A_1G^2 + 2A_1B^2 \\ &= 3PG^3 + \tfrac{1}{3}(BC^2 + CA^2 + AB^2). \end{aligned}$$

$$\begin{aligned} \text{Similarly}\; PA_1^2 + PB_1^2 + PC_1^2 &= 3PG^2 + \tfrac{1}{3}(B_1C_1^2 + C_1A_1^2 + A_1B_1^2) \\ &= 3PG^2 + \frac{1}{3\cdot 4}(BC^2 + CA^2 + AB^2) \end{aligned}$$

$$\cdots\cdots\cdots\cdots = \cdots\cdots\cdots\cdots\cdots\cdots$$

$$\text{Hence}\quad PA_n^2 + PB_n^2 + PC_n^2 = 3PG^2 + \frac{1}{3\cdot 4^n}(BC^2 + CA^2 + AB^2)$$

(30) *If the sides of triangle ABC be divided at A_1, B_1 C_1, so that $BA_1 : BC = CB_1 : CA = AC_1 : AB = m$, then*

$$A_1B_1C_1 = ABC\{1 - 3m(1 - m)\}.$$

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* Mr E. Conolly in *Mathematical Questions from the Educational Times*, IV. 76 (1865).

† Mr Stephen Watson, in *Mathematical Questions from the Educational Times*, XX. 109-112 (1873), where four solutions are given. The solution in the text is Mr Watson's.

FIGURE 11.

$$
\begin{array}{ll}
\text{For} & A_1B_1C_1 = ABC - AB_1C_1 - BC_1A_1 - CA_1B_1. \\[2ex]
\text{Now} & \dfrac{AB_1C_1}{ABC} = \dfrac{AC_1}{AB} \cdot \dfrac{AB_1}{AC} = m(1-m); \\[2ex]
\text{therefore} & AB_1C_1 = ABC \cdot m(1-m). \\
\text{Similarly} & BC_1A_1 = ABC \cdot m(1-m), \\
\text{and} & CA_1B_1 = ABC \cdot m(1-m); \\
\text{therefore} & A_1B_1C_1 = ABC\{1 - 3m(1-m)\}.
\end{array}
$$

(31) Let there be a series of triangles

$$A_1B_1C_1,\ A_2B_2C_2,\ \dots\ A_nB_nC_n$$

such that each is derived from the preceding in the same way as $A_1B_1C_1$ was derived from ABC; and let them be denoted by $\triangle_1, \triangle_2, \dots \triangle_n$.

Then the formula

$$\triangle_1 = \triangle\ \{1 - 3m(1-m)\}$$

may be applied to triangle $A_2B_2C_2$;

$$
\begin{array}{ll}
\text{therefore} & \triangle_2 = \triangle_1\{1 - 3m(1-m)\} \\
 & = \triangle\ \{1 - 3m(1-m)\}^2. \\
\text{Similarly} & \triangle_3 = \triangle\ \{1 - 3m(1-m)\}^3, \\
\text{and} & \triangle_n = \triangle\ \{1 - 3m(1-m)\}^n.
\end{array}
$$

Hence $\triangle, \triangle_1, \triangle_2 \dots$ form a decreasing geometrical progression, whose sum to infinity is equal to $\dfrac{\triangle}{3m(1-m)}$.

(32) This sum is a minimum when the product $m(1-m)$ is a maximum, that is, when $m = \frac{1}{2}$. Hence the minimum sum is $\frac{4}{3}\triangle$, and each triangle is then the complementary triangle of its predecessor.

When m is 0 or 1, the sum becomes infinite. This arises from the fact that then $\triangle_1, \triangle_2, \dots$ coincide with $\triangle$.

When m varies from 0 to 1, the sum diminishes from infinity to its maximum $\frac{4}{3}\triangle$, and then increases to infinity.

(33) *The centroid G of ABC is the centroid of* $A_1B_1C_1$, $A_2B_2C_2$, … .

FIGURE 11.

Through B_1, C_1 draw parallels to AB, AC; these parallels will intersect on BC at a point D such that

$$BD : DC = AB_1 : B_1C = BC_1 : C_1A.$$

Hence $BD = CA_1$, and AD, B_1C_1, the diagonals of the parallelogram AB_1DC_1, bisect each other at E. Now if A′ be the mid point of BC, it will also be the mid point of DA_1;
therefore AA′, A_1E, two medians of triangle ADA_1, intersect at a point G such that

$$AG = 2A'G \text{ and } A_1G = 2EG.$$

Hence since AA′, A_1E are medians of ABC, $A_1B_1C_1$ these two triangles have the same centroid G.

What has been proved with regard to ABC, $A_1B_1C_1$ will hold equally with regard to $A_1B_1C_1$, $A_2B_2C_2$; and so on. Therefore the whole series of triangles have the same centroid.*

The last property may also be proved thus †:—

FIGURE 12.

Bisect BC and A_1B_1 at A′ and F;
join AA′, C_1F cutting each other at G;
and draw B_1D parallel to AB.

Then $\quad BA_1 : CA_1 = CB_1 : AB_1,$
$\qquad\qquad = CD : BD;$
therefore $\quad BA_1 = CD;$
therefore A′ is the mid point of A_1D;
therefore A′F is parallel to DB_1, and equal to $\frac{1}{2}DB_1$.

Again $\quad B_1D : AB = CB_1 : CA,$
$\qquad\qquad = AC_1 : AB;$
therefore $\quad B_1D = AC_1;$
therefore A′F is half of AC_1, and it is parallel to it;
therefore $AG = 2A'G$ and $C_1G = 2FG$;
therefore G is the centroid of both ABC and $A_1B_1C_1$.

* The theorem that $A_1B_1C_1$ has the same centroid as ABC will be found in Pappus's *Mathematical Collection*, VIII. 2. Chasles has some remarks on the theorem in his *Aperçu historique*, 2nd ed., p. 44.

† This mode of proof was communicated to me by Mr A. J. Pressland. Compare also Fuhrmann's *Synthetische Beweise planimetrischer Sätze*, pp. 48-9 (1890).

Relations which Exist Between a Triangle and the Triangle whose Sides are the Medians of the Former.*

(34) *If ABC be any triangle, another triangle can always be constructed whose sides are equal to the medians of ABC.*

Figure 13.

Let AA′, BB′, CC′ be the medians of ABC.

Through A′ draw A′L parallel to BB′, and produce it so that A″L = A′L; join A″B′, A′C′.

Because A′ is the mid point of BC, and A′L is parallel to BB′, therefore L is the mid point of B′C.

Hence B′A′CA″ is a parallelogram, as well as B′BA′A″,

and A′A″ = BB′.

Since	A B′	is equal and parallel to C′A′
and	B′A″	„ „ „ „ „ A′C;
therefore	A″A	„ „ „ „ „ C C′;

that is AA′A″ is the triangle required.

(35) *The sides of AA′A″ are parallel to the medians of ABC*
and „ „ „ ABC „ „ „ „ „ „ AA′A″.

The first part of the theorem has been already proved.

Since AB′ : B′L = 2 : 1

therefore B′ is the centroid of AA′A″,

Now the median B′A″ is parallel to BC,

„ „ B′A coincides with CA,

and „ „ B′A′ is parallel to AB.

(36) A″, B′, C′ are collinear.

(37) *If ABC be a triangle whose centroid is G, DEF the triangle whose sides are the medians of ABC, that is EF = AA′, FD = BB′, DE = CC′, then*

$$\angle D = GBC + GCB, \quad \angle E = GCA + GAC, \quad \angle F = GAB + GBA.$$

* In connection with this subject, the following authorities may be consulted:

Gergonne's *Annales*, II. 93 (1811).

Supplemente zu G. S. Klügel's Wörterbuche der reinen Mathematik, Vol. I. Art. "Dreieck" (J. A. Grunert), p. 706 (1833).

Nouvelles Annales, III. 457-460 (1844).

Battaglini's *Giornale di Matematiche*, I. 126-7 (1863).

Grunert's *Archiv*, XLI. 112-4 (1864).

FIGURE 13.

The angles which are equal have been marked with the same number; and the triangle DEF corresponds to the triangle A″AA′.

(38) *If* $ABC,\ A_1B_1C_1,\ A_2B_2C_2\ \ldots\ldots$
$DEF,\ D_1E_1F_1,\ D_2E_2F_2\ \ldots\ldots$

be two sets of triangles such that the sides of

DEF *are equal to the medians of* ABC
$A_1B_1C_1$,, ,, ,, ,, ,, ,, DEF
$D_1E_1F_1$,, ,, ,, ,, ,, ,, $A_1B_1C_1$,

and so on;

the triangles $ABC,\ A_1B_1C_1,\ \ldots$ *will be similar to each other* *
and $DEF,\ D_1E_1F_1\ \ldots$,, ,, ,, ,, ,, ,,

FIGURE 14.

The proof of the theorem will appear from the figure † if it be observed that

Triangles	correspond to	
D E F	A″A A′	(4, 5; 6, 1; 2, 3),
$A_1B_1C_1$	L A‴A	(1, 2; 3, 4; 5, 6),
$D_1E_1F_1$	A M A^{IV}	(4, 5; 6, 1; 2, 3),
$A_2B_2C_2$	A^{V}A N	(1, 2; 3, 4; 5, 6),
$D_2E_2F_2$	P A^{VI}A	(4, 5; 6, 1; 2, 3).

The theorem may be proved also as follows:‡

If $m_1,\ m_2\ m_3$, be the three medians of ABC, then

$$m_1^2 = \tfrac{1}{2}(b^2 + c^2 - \tfrac{1}{2}a^2),\quad m_2^2 = \tfrac{1}{2}(c^2 + a^2 - \tfrac{1}{2}b^2),\quad m_3^2 = \tfrac{1}{2}(a^2 + b^2 - \tfrac{1}{2}c^2).$$

Make a triangle whose sides are $m_1,\ m_2,\ m_3$,
and let its medians be $a_1,\ b_1,\ c_1$; then

$$a_1^2 = \tfrac{1}{2}(m_2^2 + m_3^2 - \tfrac{1}{2}m_1^2),\ b_1^2 = \tfrac{1}{2}(m_3^2 + m_1^2 - \tfrac{1}{2}m_2^2),\ c_1^2 = \tfrac{1}{2}(m_1^2 + m_2^2 - \tfrac{1}{2}m_3^2)$$
$$= \tfrac{9}{16}a^2\ ,\qquad = \tfrac{9}{16}b^2\ ,\qquad = \tfrac{9}{16}c^2\,;$$

therefore $a_1 = \tfrac{3}{4}a,\ b_1 = \tfrac{3}{4}b,\ c_1 = \tfrac{3}{4}c,$

and $a_1 : b_1 : c_1 = a : b : c.$

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* Gergonne's *Annales*, II. 93 (1811).

† The figure has been taken from Grunert's article "Dreieck" previously referred to.

‡ Grunert's *Archiv*, XLI. 112-4 (1864).

(39) If

$$\triangle,\ \triangle_2,\ \triangle_4\ \ldots\ldots$$
$$\triangle_1,\ \triangle_3,\ \triangle_5\ \ldots\ldots$$

denote the two sets of triangles in (38), the sides of

$\triangle$	are	a,	b,	c
$\triangle_2$	,,	$\frac{3}{4}a$,	$\frac{3}{4}b$,	$\frac{3}{4}c$
$\triangle_4$	,,	$(\frac{3}{4})^2a$,	$(\frac{3}{4})^2b$,	$(\frac{3}{4})^2c$
$\triangle_{2n}$	,,	$(\frac{3}{4})^na$,	$(\frac{3}{4})^nb$,	$(\frac{3}{4})^nc$
$\triangle_1$	,,	m_1,	m_2,	m_3
$\triangle_3$	,,	$\frac{3}{4}m_1$,	$\frac{3}{4}m_2$,	$\frac{3}{4}m_3$
$\triangle_5$	,,	$(\frac{3}{4})^2m_1$,	$(\frac{3}{4})^2m_2$,	$(\frac{3}{4})^2m_3$
$\triangle_{2n+1}$	,,	$(\frac{3}{4})^nm_1$,	$(\frac{3}{4})^nm_2$,	$(\frac{3}{4})^nm_3$

(40) *The triangles* $\triangle,\ \triangle_1,\ \triangle_2,\ \triangle_3\ \ldots$ *form a geometrical progression** *whose common ratio is* $\frac{3}{4}$.

FIGURE 13.

Since	$AL = \frac{3}{4}AC$
therefore	$AA'L = \frac{3}{4}AA'C$;
therefore	$AA'A'' = \frac{3}{4}ABC$; and so on.

(41) $$\triangle + \triangle_1 + \triangle_2 + \ldots ad\ infinitum = 4\triangle.$$

(42) If $p,\ p_2,\ p_4\ \ldots$ be the perimeters of $\triangle,\ \triangle_2,\ \triangle_4\ldots$
$$p + p_2 + p_4 + \ldots ad\ infinitum = 4p.$$

(43) If $p_1,\ p_3,\ p_5\ \ldots$ be the perimeters of $\triangle_1,\ \triangle_3,\ \triangle_5\ldots$
$$p_1 + p_3 + p_5 + \ldots ad\ infinitum = 4p_1.$$

(44) $$\triangle + \triangle_2 + \triangle_4 + \ldots ad\ infinitum = \tfrac{16}{7}\triangle.$$

(45) $$\triangle_1 + \triangle_3 + \triangle_5 + \ldots ad\ infinitum = \tfrac{12}{7}\triangle.$$

(46) If G be the centroid of ABC and another triangle $A_0B_0C_0$ be formed with sides respectively equal to $\sqrt{3}\,GA$, $\sqrt{3}\,GB$, $\sqrt{3}\,GC$, then ABC may be derived from $A_0B_0C_0$ in the same way as the latter was derived from the former, that is, the relation between the triangles is a conjugate one.†

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* Gergonne's *Annales*, II. 93 (1811).

† Rev. T. C. Simmons in Milne's *Companion to the Weekly Problem Papers*, pp. 150-1 (1888).

(47) The areas of ABC, $A_0B_0C_0$ are equal.*

These two theorems follow without much difficulty from what precedes.

(48) *If through the centroid G of a triangle ABC a straight line be drawn cutting BC, CA, AB in D, E, F and the points E, F be on the same side of G then*†

$$\frac{1}{GE}+\frac{1}{GF}=\frac{1}{GD}.$$

FIGURE 15.

Through A draw AN parallel to BC meeting DEF in K, and through G draw LMN parallel to AB meeting BC, CA, AN in L, M, N.

Then $\qquad$ LG = MG, and CM = 2AM.

But since triangles CML, AMN are similar,

therefore $\qquad$ ML = 2MN;

therefore $\qquad$ GM = MN.

Hence AG, AM, AN, AF form a harmonic pencil;

and they are cut by the transversal GEKF;

therefore G, E, K, F form a harmonic range;

therefore
$$\frac{1}{GE}+\frac{1}{GF}=\frac{2}{GK}$$
$$=\frac{1}{GD}$$

since $\qquad$ GK = 2GD.

* Rev. T. C. Simmons in Milne's *Companion to the Weekly Problem Papers*, p. 151 (1888).

† This property, proved in the manner given, will be found in Maclaurin's *Algebra* (1748) in the Appendix, *De Linearum Geometricarum Proprietatibus generalibus Tractatus*, §98 or p. 57. A proof by Dr E. v. Hunyady of Pesth, by means of transversals, will be found in Schlömilch's *Zeitschrift*, VII. 268-9 (1862).

Formulæ connected with the Medians.

The medians in terms of the sides.

$$\left.\begin{aligned} 4m_1^2 &= -a^2 + 2b^2 + 2c^2 \\ 4m_2^2 &= 2a^2 - b^2 + 2c^2 \\ 4m_3^2 &= 2a^2 + 2b^2 - c^2 \end{aligned}\right\} \qquad (1)$$

$$\left.\begin{aligned} 4m_1^2 &= -a^2 + (b+c)^2 + (b-c)^2 \\ 4m_2^2 &= -b^2 + (c+a)^2 + (c-a)^2 \\ 4m_3^2 &= -c^2 + (a+b)^2 + (a-b)^2 \end{aligned}\right\} \qquad (2)$$

$$\left.\begin{aligned} (2m_1 + b - c)(2m_1 - b + c) &= (a+b+c)(-a+b+c) \\ (2m_2 + c - a)(2m_2 - c + a) &= (a+b+c)(a-b+c) \\ (2m_3 + a - b)(2m_3 - a + b) &= (a+b+c)(a+b-c) \end{aligned}\right\} \qquad (3)$$

$$\left.\begin{aligned} (b + c + 2m_1)(b + c - 2m_1) &= (a+b-c)(a-b+c) \\ (c + a + 2m_2)(c + a - 2m_2) &= (-a+b+c)(a+b-c) \\ (a + b + 2m_3)(a + b - 2m_3) &= (a-b+c)(-a+b+c) \end{aligned}\right\} \qquad (4)$$

$$4(m_1^2 + m_2^2 + m_3^2) = 3(a^2 + b^2 + c^2) \qquad (5)$$

$$3(\mathrm{AG}^2 + \mathrm{BG}^2 + \mathrm{CG}^2) = a^2 + b^2 + \mathrm{c}^2 \qquad (6)$$

$$12(\mathrm{A'G}^2 + \mathrm{B'G}^2 + \mathrm{C'G}^2) = a^2 + b^2 + c^2 \qquad (7)$$

$$m_1 \cdot \mathrm{A\,G} + m_2 \cdot \mathrm{B\,G} + m_3 \cdot \mathrm{C\,G} = \tfrac{1}{2}(a^2 + b^2 + c^2) \qquad (8)$$

$$m_1 \cdot \mathrm{A'G} + m_2 \cdot \mathrm{B'G} + m_3 \cdot \mathrm{C'G} = \tfrac{1}{4}(a^2 + b^2 + c^2) \qquad (9)$$

$$16(m_1^4 + m_2^4 + m_3^4) = 9(a^4 + b^4 + c^4) \qquad (10)$$

$$16(m_2^2 m_3^2 + m_3^2 m_1^2 + m_1^2 m_2^2) = 9(b^2c^2 + c^2a^2 + a^2b^2) \qquad (11)$$

The sides in terms of the medians.

$$\left.\begin{aligned} 9a^2 &= -4m_1^2 + 8m_2^2 + 8m_3^2 \\ 9b^2 &= 8m_1^2 - 4m_2^2 + 8m_3^2 \\ 9c^2 &= 8m_1^2 + 8m_2^2 - 4m_3^2 \end{aligned}\right\} \qquad (12)$$

$$\left.\begin{aligned} \tfrac{9}{4}a^2 &= -m_1^2 + (m_2 + m_3)^2 + (m_2 - m_3)^2 \\ \tfrac{9}{4}b^2 &= -m_2^2 + (m_3 + m_1)^2 + (m_3 - m_1)^2 \\ \tfrac{9}{4}c^2 &= -m_3^2 + (m_1 + m_2)^2 + (m_1 - m_2)^2 \end{aligned}\right\} \qquad (13)$$

$$\left.\begin{aligned} (\tfrac{3}{2}a + m_2 - m_3)(\tfrac{3}{2}a - m_2 + m_3) &= (m_1 + m_2 + m_3)(-m_1 + m_2 + m_3) \\ (\tfrac{3}{2}b + m_3 - m_1)(\tfrac{3}{2}b - m_3 + m_1) &= (m_1 + m_2 + m_3)(m_1 - m_2 + m_3) \\ (\tfrac{3}{2}c + m_1 - m_2)(\tfrac{3}{2}c - m_1 + m_2) &= (m_1 + m_2 + m_3)(m_1 + m_2 - m_3) \end{aligned}\right\} \qquad (14)$$

$$\left.\begin{aligned} (m_2 + m_3 + \tfrac{3}{2}a)(m_2 + m - \tfrac{3}{2}a) &= (m_1 + m_2 + m_3)(m_1 - m_2 + m_3) \\ (m_3 + m_1 + \tfrac{3}{2}b)(m_3 + m_1 - \tfrac{3}{2}b) &= (m_1 + m_2 + m_3)(m_1 + m_2 - m_3) \\ (m_1 + m_2 + \tfrac{3}{2}c)(m_1 + m_2 - \tfrac{3}{2}c) &= (-m_1 - m_2 + m_3)(-m_1 + m_2 + m_3) \end{aligned}\right\} \qquad (15)$$

$$\left.\begin{aligned}3(b^2 \sim c^2) &= 4(m_3^2 \sim m_2^2)\\3(c^2 \sim a^2) &= 4(m_1^2 \sim m_3^2)\\3(a^2 \sim b^2) &= 4(m_2^2 \sim m_1^2)\end{aligned}\right\} \qquad (16)$$

If
$$\begin{aligned}2s &= a+b+c\\2s' &= b+c+2m_1\\2s'' &= c+a+2m_2\\2s''' &= a+b+2m_3\end{aligned}$$

then (3) and (4) become

$$\left.\begin{aligned}(s'-b)(s'-c) &= s(s-a) & s'(s'-2m_1) &= (s-b)(s-c)\\(s''-c)(s''-a) &= s(s-b) & s''(s''-2m_2) &= (s-c)(s-a)\\(s'''-a)(s'''-b) &= s(s-c) & s'''(s'''-2m_3) &= (s-a)(s-b)\end{aligned}\right\} \qquad (17)$$

$$\left.\begin{aligned}\Delta^2 &= s'(s'-b)(s'-c)(s'-2m_1)\\&= s''(s''-c)(s''-a)(s''-2m_2)\\&= s'''(s'''-a)(s'''-b)(s'''-2m_3)\end{aligned}\right\} \qquad (18)$$

For each $= s(s-a)(s-b)(s-c)$

If
$$\begin{aligned}2m &= m_1+m_2+m_3\\2n_1 &= m_2+m_3+\tfrac{3}{2}a\\2n_2 &= m_3+m_1+\tfrac{3}{2}b\\2n_3 &= m_1+m_2+\tfrac{3}{2}c\end{aligned}$$

then (14) and (15) become

$$\left.\begin{aligned}(n_1-m_2)(n_1-m_3) &= m(m-m_1) & n_1(n_1-\tfrac{3}{2}a) &= (m-m_2)(m-m_3)\\(n_2-m_3)(n_2-m_1) &= m(m-m_3) & n_2(n_2-\tfrac{3}{2}b) &= (m-m_3)(m-m_1)\\(n_3-m_1)(n_3-m_2) &= m(m-m_3) & n_3(n_3-\tfrac{3}{2}c) &= (m-m_1)(m-m_2)\end{aligned}\right\} \qquad (19)$$

Area of triangle in terms of its medians.

FIGURE 1.

Because $\quad ABC = 2ABA' = 6GBA' = 3GBL;$

and $\quad GL = \tfrac{2}{3}m_1,\ GB = \tfrac{2}{3}m_2,\ BL = \tfrac{2}{3}m_3;$

therefore $\quad \Delta^2 = 9(GBL)^2$

$$= 9\left\{\frac{m_1+m_2+m_3}{3}\cdot\frac{-m_1+m_2+m_3}{3}\cdot\frac{m_1-m_2+m_3}{3}\cdot\frac{m_1+m_2-m_3}{3}\right\}$$

$$= \tfrac{1}{9}\{(m_1+m_2+m_3)(-m_1+m_2+m_3)(m_1-m_2+m_3)(m_1+m_2-m_3)\}.$$

Let $\quad 2m = m_1+m_2+m_3$

then $\quad \Delta^2 = \tfrac{16}{9}\{m(m-m_1)(m-m_2)(m-m_3)\} \qquad (20)$

If
$$2m' = -m_1 + m_2 + m_3$$
$$2m'' = m_1 - m_2 + m_3$$
$$2m''' = m_1 + m_2 - m_3$$
then
$$\Delta = \tfrac{4}{3}\sqrt{mm'm''m'''} \qquad (21)$$

$$\left.\begin{aligned}\Delta &= \tfrac{4}{3}\sqrt{n_1(n_1-m_2)(n_1-m_3)(n_1-\tfrac{3}{2}a)}\\ &= \tfrac{4}{3}\sqrt{n_2(n_2-m_3)(n_2-m_1)(n_2-\tfrac{3}{2}b)}\\ &= \tfrac{4}{3}\sqrt{n_3(n_3-m_1)(n_3-m_2)(n_3-\tfrac{3}{2}c)}\end{aligned}\right\} \qquad (22)$$

This is deduced from (20) by means of (19).

If R, S, T be the projections of G on the sides

$$\left.\begin{aligned}\mathrm{BR} &= \frac{3a^2 - b^2 + c^2}{6a} & \mathrm{CR} &= \frac{3a^2 + b^2 - c^2}{6a}\\ \mathrm{CS} &= \frac{a^2 + 3b^2 - c^2}{6b} & \mathrm{AS} &= \frac{-a^2 + 3b^2 + c^2}{6b}\\ \mathrm{AT} &= \frac{-a^2 + b^2 + 3c^2}{6c} & \mathrm{BT} &= \frac{a^2 - b^2 + 3c^2}{6c}\end{aligned}\right\} \qquad (23)$$

$$\mathrm{ST} = \frac{4m_1\Delta}{3bc} \qquad \mathrm{TR} = \frac{4m_2\Delta}{3ca} \qquad \mathrm{RS} = \frac{4m_3\Delta}{3ab} \qquad (24)$$

$$\mathrm{ST} : \mathrm{TR} : \mathrm{RS} = am_1 : bm_2 : cm_3 \qquad (25)$$

Of the preceding formulæ, (8) and (9) are given by C. F. A. Jacobi, *De Triangulorum Rectilineorum Proprietatibus*, p. 7 (1825); (10) and (11) occur in Hind's *Trigonometry*, 4th ed., p. 244 (1841); (12) in Thomas Simpson's *Select Exercises*, Part II., Problem xxii. (1752); (2)-(4), (13)-(19), (21), (22) are due to Thomas Weddle. See *Lady's and Gentleman's Diary* for 1848, pp. 74-75. I have changed the notation adopted by Weddle.

On the authority of Férussac's *Bulletin des Sciences Mathématiques*, xii. 297 (1829), formula (20) should be assigned to Professor Desgranges.

§ 2. Circumcentre.

*The perpendiculars to the sides of a triangle from the mid points of the sides are concurrent.**

The following demonstration † may be compared with the demonstration of § 5.

Figure 16.

Let A′, B′, C′ be the mid points of BC, CA, AB,
and let A′X′, B′Y′, C′Z′ be perpendicular to BC, CA, AB.

Join	B′C′, C′A′, A′B′.
Then	B′C′ is parallel to BC;
therefore	A′X′ is perpendicular to B′C′.
Hence	B′Y′ „ „ „ C′A′
and	C′Z′ „ „ „ A′B′.

If therefore it be assumed as true that the perpendiculars to the sides of a triangle from the opposite vertices are concurrent,

A′X′, B′Y′, C′Z′ are concurrent.

Another demonstration is obtained at once from the theorem, ‡
If three points be taken on the sides of a triangle such that the sums of the squares of the alternate segments taken cyclically are equal, the perpendiculars to the sides of the triangle at these points are concurrent.

The point of concurrency, which will be denoted by O, is the centre of the circle circumscribed about ABC. This circle is often called the *circumcircle*,§ and the centre of it the *circumcentre*.§

The radius of the circumcircle is usually denoted by R.

(1) The circumcentre of a triangle is the orthocentre of its complementary triangle.

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* Euclid's *Elements*, IV. 5.

† C. Adams, *Die Lehre von den Transversalen*, p. 21 (1843).

‡ F. G. de Oppel, *Analysis Triangulorum*, p. 32 (1746).

§ These terms as well as *incircle, excircle, midcircle, incentre, excentre, midcentre* were suggested by W. H. H. H[udson]. See *Nature*, XXVIII. 7, 104 (1883). The terms *Umkreis, Inkreis, Ankreis, Mittenkreis* have been more or less in use in Germany since 1866, as may be seen from Schlömilch's *Zeitschrift*.

The perpendiculars to the sides of a triangle from the mid points of the sides are sometimes called *médiatrices* in France and Belgium.

(2) Since the complementary and the fundamental triangles are similar, and since their sides are in the ratio of 1 to 2, the distance of the circumcentre of a triangle from any side is half of the distance between the orthocentre of the triangle and the vertex opposite that side.*

(3) If O be the circumcentre of a triangle ABC, the circle on OA as diameter bisects AB and AC.

Similarly for the circles on OB and OC.

(4) *If the circle on OA as diameter should cut BC at P and P′, then AP is a mean proportional* † *between BP and CP, and AP′ is a mean proportional between BP′ and CP′.*

FIGURE 17.

Produce AP to meet the circumcircle at Q.

The circle on OA as diameter touches the circumcircle at A; therefore A is the homothetic centre of the two circles;

therefore $\quad AP : AQ = 1 : 2$;

therefore $\quad BP \cdot CP = AP \cdot QP = AP^2$.

(5) *By the following construction* ‡ *a point P will be found in the base BC of ABC such that the ratio* $AP^2 : BP \cdot CP$ *has a given value.*

FIGURE 18.

Join AO, and divide it at L so that AL : LO has the given value; then the circle with centre L and radius LA will meet BC in two points P, P′ satisfying the condition.

Produce AP to meet the circumcircle in Q.

Then
$$\begin{aligned} AP : PQ &= AP^2 : AP \cdot PQ \\ &= AP^2 : BP \cdot PC \\ &= AL : LO \end{aligned};$$

therefore LP is parallel to OQ;
therefore LP = LA, since OQ = OA.

---

* *Ladies' Diary* for 1785.

† Given without proof in the *Ladies' Diary* for 1759.

‡ Rev. J. Wolstenholme in the *Educational Times*, XXIX., 273 (1877). Four solutions are given in *Mathematical Questions from the Educational Times*, XXVII, 63-66 (1877); the one in the text is the last.

If AP′Q′ be the other position of APQ,

then $$AP : PQ = AP' : P'Q' ;$$

therefore QQ′ is parallel to BC ;
therefore arc BQ = arc CQ′,
and AP, AP′ are isogonal with respect to $\angle$ A.

(6) If from a point O within or without a triangle ABC, perpendiculars OD, OE, OF are drawn to the sides BC, CA, AB, and circles are circumscribed about the triangles OEF, OFD, ODE ; the area of the triangle formed by joining the centres of these circles is one-fourth of the area * of the triangle ABC

FIGURE 19.

The centres of these circles are the mid points of OA, OB, OC.

(7) *If from a point O within triangle ABC perpendiculars OD, OE, OF be drawn to BC, CA, AB, then* †

$$2R(EF + FD + DE) = OA \cdot BC + OB \cdot CA + OC \cdot AB.$$

FIGURE 19.

For A, F, O, E lie on the circle whose diameter is OA, and the chord EF subtends the same angle A at the circumference of this circle as BC does at that of the circumcircle of ABC ;

therefore $$EF : OA = BC : 2R ;$$

therefore $$2R \cdot EF = OA \cdot BC.$$

(8) If O be on that arc of the circumcircle on which angle C stands, then,† by Ptolemy's theorem,

$$OA \cdot BC + OB \cdot CA - OC \cdot AB = 0 ;$$

therefore $$EF + FD - DE \qquad = 0 ;$$

therefore D, E, F are collinear,
which is another proof of the property of the Wallace line.

---

* Todhunter's *Plane Trigonometry*, Chap. XVI., Ex. 41 (1859).

† Both (7) and (8) are due to Mr E. M. Langley, who applies the first of them to the problem of finding the triangle of minimum perimeter inscribed in a given triangle, and to the determination of the trilinear co-ordinates of the Brocard points. See *Sixteenth General Report of the Association for the Improvement of Geometrical Teaching*, pp. 34-5 (1890).

(9) *If O be the circumcentre of ABC, and AO, BO, CO be produced to meet the circumcircle in U, V, W, the triangle UVW is congruent to ABC.*

FIGURE 20.

For $\angle AUV = \angle ABO = \angle BAO$,

$\angle AUW = \angle ACO = \angle CAO$;

therefore $\angle VUW = \angle BAC$;

therefore UVW is similar to ABC.

But these two triangles are inscribed in the same circle;

therefore they are congruent.

(10) The figures BCVW, CAWU, ABUV are rectangles.

(11) If ABC be a triangle, and BW, CV be perpendicular to BC; CU, AW perpendicular to CA; AV, BU perpendicular to AB, the three straight lines AU, BV, CW are concurrent at the circumcentre of ABC, and the six points A, B, C, U, V, W are concyclic.*

FIGURE 20.

(12) *Triangles $A_1B_1C_1$, $A_2B_2C_2$ circumscribed about ABC in such a manner that their sides are perpendicular to those of ABC are congruent* † *to each other and similar to ABC.*

FIGURE 21.

Let C_1A_1, A_2B_2; A_1B_1, B_2C_2; B_1C_1, C_2A_2

meet at U, V, W.

Then BB_1VB_2 is a parallelogram;

therefore $BB_1 = B_2V$.

But $BW = CV$;

therefore $B_1W = CB_2$.

Again WC_1CC_2 is a parallelogram;

therefore $WC_1 = C_2C$;

therefore $B_1C_1 = B_1C_2$.

Similarly $C_1A_1 = C_2A_2$, $A_1B_1 = A_2B_2$.

---

* C. F. A. Jacobi, *De Triangulorum Rectilineorum Proprietatibus*, p. 56 (1825).

† The first part of the theorem is given by Jacobi, p. 56.

Lastly $\angle BAC = 90^\circ - \angle BAA$

$= \angle B_1AA_2 = \angle A$.

Similarly $\angle ABC = \angle B_1$, $\angle ACB = C_1$.

This theorem is a particular case of a more general one.

(13) *The three lines A_1A_2, B_1B_2, C_1C_2 are concurrent* at the circumcentre of ABC.*

FIGURE 21.

For AU, BV, CW are concurrent at O, the circumcentre of ABC; and O is the mid point of AU, BV, CW.
Now since AA_1UA_2 is a parallelogram,
therefore A_1A_2 passes through the mid point of AU.

Similarly for B_1B_2, C_1C_2.

(14) *If a point P be taken inside the triangle ABC, and circles be circumscribed about the triangles PBC, PCA, PAB, and their centres O_1, O_2, O_3 be joined, the angles of triangle $O_1O_2O_3$ are supplementary to the angles BPC, CPA, APB.*

FIGURE 22.

For O_2O_3, O_3O_1, O_1O_2 are respectively perpendicular to PA, PB, PC.

(15) *If through A any straight line MN be drawn meeting the circumferences PCA, PAB in M, N, then MC, NB will intersect on the circumference† PBC.*

Let MC, NB intersect at L.

Then $\angle M = 180^\circ - \angle CPA$,

$\angle N = 180^\circ - \angle APB$;

therefore $\angle M + \angle N = 360^\circ - (\angle CPA + \angle APB)$,

$= \angle BPC$;

therefore $\angle L = 180^\circ - \angle BPC$;

therefore L is on the circumference PBC.

---

* Jacobi does not state this property, but from the way in which he letters the figures of theorems (11) and (12) it is probable that he knew it. The property is explicitly stated, along with some others, by Mr Lemoine in his paper read at the Lyons meeting (1873) of the *Association Française pour l'avancement des Sciences.*

† Rochat in Gergonne's *Annales*, II. 29 (1811). To him also are due (17) and (19).

(16) If L be any point on the circumference PBC, and if LC, LB meet the circumferences PCA, PAB again in M, N, then M, A, N are collinear.

(17) Triangle LMN is similar to $O_1O_2O_3$.

If the point P be fixed, the triangles $O_1O_2O_3$, LMN are given in species.

(18) *The angles which MN, NL, LM make with AP, BP, CP respectively are equal.*

For $\angle PAN = 180° - \angle PBN = \angle PBL$
$= 180° - \angle PCL = \angle PCM.$

(19) *Of all the triangles such as LMN whose sides pass through A, B, C, and whose vertices are situated on the circles O_1, O_2, O_3, that triangle L'M'N' is a maximum whose sides are perpendicular to AP, BP, CP.*

FIGURE 22.

For triangles L'M'N', LMN are similar,
and PL', PL are corresponding lines in these triangles.
Now PL' is a diameter of the circle O_1;
therefore PL' is greater than PL;
therefore L'M'N' is greater than LMN.

(20) If O be the circumcentre of ABC, and about the triangles OBC, OCA, OAB circles be circumscribed whose centres are O_1, O_2, O_3, the triangle $O_1O_2O_3$ has its angles equal to $180° - 2A$, $180° - 2B$, $180° - 2C$.

It will be found that $O_1O_2O_3$ is similar to XYZ. See § 5.

(21) *O is the incentre of the triangle $O_1O_2O_3$.*

FIGURE 22.

In the diagram suppose P to be replaced by O, and let V, W be the mid points of BO, CO.

Then the right-angled triangles OVO_1, OWO_1 have two sides of the one equal to two sides of the other;
therefore OO_1 bisects $\angle O_2O_1O_3$.
Similarly for OO_2, OO_3.

(22.) *If OO_1, OO_2, OO_3 be produced to meet the circles OBC, OCA, OAB in L', M', N', the triangle $L'M'N'$ will be circumscribed about ABC, will be similar and similarly situated to $O_1O_2O_3$, and will have O for its incentre.*

FIGURE 22.

For $$\angle OAM' + \angle OAN' = 180^\circ;$$
therefore M′, A, N′ are collinear, and M′N′ is parallel to O_2O_3.

Since OA, OB, OC are equal, and perpendicular to M′N′, N′L′, L′M′;

therefore O is the incentre of L′M′N′.

Many relations between the triangles $O_1O_2O_3$ and ABC may be derived from the relations between XYZ and ABC, seeing that $O_1O_2O_3$ is similar to XYZ and that the ratio of the radii of their in-circles is $\frac{1}{2}R : \rho$.

§ 3. Incentre.

*The internal angular bisectors of a triangle are concurrent.**

The following demonstration † is different from the usual one.

Figure 23.

Let AL be the internal bisector of $\angle$ A, and let the internal bisector of $\angle$ B cut it at I.

Then
$$AI : LI = BA : BL$$
$$= CA : CL ;$$

therefore the internal bisector of $\angle$ C passes through I.

The point of concurrency, which will be denoted by I, is the centre of the circle inscribed in ABC. This circle is often called the *incircle*,‡ and the centre of it the *incentre*.‡

The radius of the incircle is usually denoted by r.

The following method § of inscribing a circle in a given triangle will be better understood after a perusal of § 4 (5). As regards practical execution it is the simplest yet obtained.

Figure 24.

Along CA take AP equal to AB, and CQ equal to CB.

With A as centre and PQ as radius describe a circle cutting AC, AB in the points S, T.

With S as centre and PQ as radius describe a circle cutting the first circle in two points; the straight line joining these two points passes through the incentre.

With T as centre and PQ as radius describe a circle cutting the first circle in two points; the straight line joining these two points passes through the incentre.

Hence the incentre is determined as well as the radius of the incircle.

The proof will be evident from the following considerations.

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* Euclid's *Elements*, IV. 4.

† Todhunter's *Elements of Euclid*, p. 312 (1864).

‡ See the note on p. 32.

§ Mr E. Lemoine in the Report (second part) of the 21st session of the *Association Française pour l'avancement des sciences*, p. 49 (1892).

The dotted straight lines bisect AS, AT perpendicularly.

Now $$AS = AT = PQ = CP - CQ = b + c - a;$$
therefore if E, F be the mid points of AS, AT
$$AE = AF = \tfrac{1}{2}(b + c - a) = s_1$$
and E, F are points of contact of the incircle.

(1) The area of a triangle is equal to the rectangle * under the semiperimeter † of the triangle and the radius of the incircle.

This is expressed, $\Delta = sr$,
where $$s = \tfrac{1}{2}(a + b + c).$$

(2) *If P be any point inside ABC, and PA, PB, PC be denoted by* α, β, γ, *and the radii of the incircles of PBC, PCA, PAB by* ρ_1, ρ_2, ρ_3, *then*

$$(\rho_2 + \rho_3)\alpha + (\rho_3 + \rho_1)\beta + (\rho_1 + \rho_2)\gamma = (r - \rho_1)a + (r - \rho_2)b + (r - \rho_3)c.$$

For
$$\begin{aligned} 2ABC &= r\,(a + b + c), \\ 2PBC &= \rho_1(a + \beta + \gamma), \\ 2PCA &= \rho_2(b + \gamma + \alpha), \\ 2PAB &= \rho_3(c + \alpha + \beta). \end{aligned}$$

But $$ABC = PBC + PCA + PAB;$$
therefore $$\rho_1(a + \beta + \gamma) + \rho_2(b + \gamma + \alpha) + \rho_3(c + \alpha + \beta) = r(a + b + c);$$

whence the result follows.

If P be outside ABC,

$$(\rho_2 + \rho_3)\alpha + (\rho_3 - \rho_1)\beta + (-\rho_1 + \rho_2)\gamma = (r + \rho_1)a + (r - \rho_2)b + (r - \rho_3)c.$$

(3) *If I be the incentre of ABC, and AI, BI, CI be produced to meet the circumcircle in U, V. W, then the sides of UVW are perpendicular to AI, BI, CI.*

---

* This is established in the course of the proof of Heron's theorem regarding the area of a triangle. See § 8.

† The semiperimeter of a triangle is usually denoted, in this country and North America, by s; on the continent of Europe it is generally denoted by p. Euler, who was one of the first if not the first to introduce the notation a, b, c for the sides of ABC, denotes the semiperimeter $\frac{1}{2}(AB + BC + CA)$ by S. See an article by him entitled *Variæ demonstrationes geometriæ* printed in *Novi Commentarii Academiæ Scientiarum Imperialis Petropolitanæ* for the years 1747-8, I. 53 (1750).

FIGURE 25.

Join U, V, W with A, B, C.

The arc BU = CU, CV = AV, AW = BW;

therefore arc UBW = arc CU + arc AW;

therefore $\angle$ U C I = $\angle$ UIC;

therefore U I = UC = UB.

Similarly V I = VC = VA,

and WI = WA = WB.

Hence AWIV, BUIW, CVIU are kites;

therefore VW, WU, UV are perpendicular to AI, BI, CI.

(4) *The angles of UVW are respectively equal to*

$$\tfrac{1}{2}(B+C),\ \tfrac{1}{2}(C+A),\ \tfrac{1}{2}(A+B).$$

For
$$\begin{aligned}\angle \mathrm{WUV} &= \angle \mathrm{AUV} + \angle \mathrm{AUW}\\ &= \angle \mathrm{ABV} + \angle \mathrm{ACW}\\ &= \tfrac{1}{2}(\mathrm{B}+\mathrm{C}).\end{aligned}$$

Hence whatever be the size of the angles A, B, C, triangle UVW is always acute-angled.

(5) The angles of ABC expressed in terms of the angles of UVW are

$$\begin{aligned}\mathrm{A} &= -\mathrm{U}+\mathrm{V}+\mathrm{W} = 180^\circ - 2\mathrm{U}\\ \mathrm{B} &= \mathrm{U}-\mathrm{V}+\mathrm{W} = 180^\circ - 2\mathrm{V}\\ \mathrm{C} &= \mathrm{U}+\mathrm{V}-\mathrm{W} = 180^\circ - 2\mathrm{W}.\end{aligned}$$

Compare § 5, (8).

(6) $$UVW : ABC = R : 2r.$$

Join the circumcentre O with A, B, C.

Then
$$\begin{aligned}2\mathrm{UVW} &= \text{hexagon AWBUCV}\\ &= \mathrm{OBUC}+\mathrm{OCVA}+\mathrm{OAWB}\\ &= \tfrac{1}{2}(\mathrm{OU}\cdot\mathrm{BC}+\mathrm{OV}\cdot\mathrm{CA}+\mathrm{OW}\cdot\mathrm{AB})\\ &= \tfrac{1}{2}\mathrm{R}\cdot 2s = \mathrm{R}s;\end{aligned}$$

and $2\mathrm{ABC} = 2rs.$

(7) *If ABC, $A_1B_1C_1$, $A_2B_2C_2$, ... $A_nB_nC_n$ be a series of triangles all inscribed in the same circle and each of which is derived from the preceding in the same manner as UVW was derived from ABC in* (3); *then when the whole number m increases indefinitely, the*

*triangle $A_{2m}B_{2m}C_{2m}$ tends towards a limiting position $\alpha\beta\gamma$, the triangle $A_{2m+1}B_{2m+1}C_{2m+1}$ tends also towards a limiting position $\alpha'\beta'\gamma'$, the two limiting triangles $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ are equilateral, and symmetrically placed with reference to the centre of the circle.**

FIGURE 26.

In triangle $A_1B_1C_1$, the perpendicular from A_1 to the opposite side is A_1A, the diameter of the circumcircle is A_1OD;

therefore the bisector of $\angle B_1A_1C_1$ is also † the bisector of $\angle AA_1D$;

therefore the vertex A_2 is at the middle of the arc AD intercepted by $\angle OA_1A$.

Hence in general, to obtain the vertex A_{n+1} draw the diameter OA_n and the straight line $A_{n-1}A_n$; the mid point of the arc intercepted by the inscribed angle thus formed is the vertex sought.

In this manner, step by step, the vertices A_2, A_3, A_4, ... are determined, and each time the inscribed angle diminishes by half. This angle therefore tends to become zero, and the two lines $A_{n+1}A_n$ and $A_{n-1}A_n$ end by coalescing with the diameter OA_n. Now since the first of these two lines is an angular bisector and the second is the corresponding perpendicular of the triangle $A_nB_nC_n$, this triangle tends to become isosceles, that is, in the limit, $\angle B_n = \angle C_n$.

Similarly $\angle C_n = \angle A_n$ and $\angle A_n = \angle B_n$;

hence the triangle $A_nB_nC_n$ tends to become equilateral.

The inscribed angles which give the vertices of even order are quartered each time. Hence the halves AA_2, A_2A_4, ... of the arcs intercepted by these angles form the terms of a geometrical series whose ratio is $\frac{1}{4}$. Since none of the arcs AA_2, A_2A_4, ... $A_{2m-1}A_{2m}$ encroaches on the preceding, their sum represents the distance AA_{2m}. This sum is, in the limit,

$$A\alpha = \tfrac{4}{3}AA_2.$$

Thus the position of the limiting equilateral triangle $\alpha\beta\gamma$ is known.

---

* Both (7) and (8) were proposed at a competitive examination in France in 1881. For the proofs see Vuibert's *Journal*, VII. 121-3 (1883).

† See § 5, (34).

In the same way the triangles of odd order tend towards the equilateral triangle $\alpha'\beta'\gamma'$ which is such that

$$A_1\alpha' = \tfrac{4}{3}A_1A_3.$$

To prove that the equilateral triangles $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ are symmetrical with respect to the centre O, it is sufficient to prove that α and α' are diametrically opposite, or that $D\alpha = A_1\alpha'$.

Now
$$D\alpha = 2A\,A_2 - A\alpha = \tfrac{4}{3}AA_2,$$
$$A_1\alpha' = \tfrac{4}{3}A_1A_3 \qquad = \tfrac{4}{3}AA_2.$$

(8) *If the radius R be taken as unity, the product of the numbers which measure the diameters of the circles inscribed in the triangles ABC, $A_1B_1C_1 \ldots A_nB_nC_n$ tends towards a limit when n increases indefinitely.*

Let $\triangle$, $\triangle_1 \ldots \triangle_n$ denote the areas of the triangles ABC, $A_1B_1C_1 \ldots A_nB_nC_n$; and d, $d_1, \ldots d_n$ the numbers which measure the diameters of their incircles.

Then
$$\triangle_1 : \triangle = R : 2r = 1 : d;$$

therefore
$$d = \frac{\triangle}{\triangle_1}$$

Similarly
$$d_1 = \frac{\triangle_1}{\triangle_2} \quad d_2 = \frac{\triangle_2}{\triangle_3}, \ldots d_n = \frac{\triangle_n}{\triangle_{n+1}};$$

therefore
$$dd_1d_2 \ldots d_n = \frac{\triangle}{\triangle_{n+1}}.$$

Now when n increases indefinitely, $\triangle_{n+1}$ approaches the area of the equilateral triangle inscribed in a circle of radius 1, that is $3\sqrt{3}/4$; hence $dd_1d_2 \ldots d_n$ approaches the limit $4\triangle/3\sqrt{3}$.

(9) It has been seen that the series of triangles $A_1B_1C_1$, $A_2B_2C_2$, etc., deduced successively from ABC and from each other can be continued indefinitely far. Can this series be extended backwards indefinitely far, and if not, when will it stop? To answer the question a solution must be found for the problem:

Given a triangle ABC inscribed in a circle, construct another inscribed triangle RST such that A, B, C shall be the mid points of the arcs ST, TR, RS.

From § 3, (4) it appears that whatever be the size of the angles R, S, T, the triangle ABC must be acute-angled. This being granted, draw the perpendiculars AX, BY, CZ of ABC, and produce them to meet the circumcircle in R, S, T. These are the vertices of the triangle sought.

The demonstration follows from the fact that H is the incentre of triangle RST. See § 5 (15).

By operating in a similar manner on RST, etc., the series may be continued backwards. It is plain, however, that as soon as a triangle is reached which is not acute-angled, the process comes to an end.

It may happen that a triangle is reached which has one angle right. Let RST be this triangle, R the right angle.

Draw RR_1 perpendicular to ST. Then the triangle antecedent to RST is the straight line R_1R, which may be considered as a triangle R_1RR. The side RR of this triangle is infinitely small and its direction is the tangent at A.

(10) Each triangle of the series considered in (7) has its angles equal to half the sum of the angles taken two and two of the preceding triangle. Consider a series of triangles such that each has its sides equal to half the sum of the sides taken two and two of the preceding triangle.

Starting with triangle ABC whose sides are a, b, c, the triangle $A_1B_1C_1$ is to be constructed whose sides are $\frac{1}{2}(b+c)$, $\frac{1}{2}(c+a)$, $\frac{1}{2}(a+b)$.

This second triangle is always possible even when a, b, c are taken at random, provided they be positive. For

$$\frac{c+a}{2} + \frac{a+b}{2} > \frac{b+c}{2}$$

$$\frac{a+b}{2} + \frac{b+c}{2} > \frac{c+a}{2}$$

$$\frac{b+c}{2} + \frac{c+a}{2} > \frac{a+b}{2}$$

FIGURE 1.

Bisect the sides of ABC at A′, B′, C′. The angular contours A′CB′, B′AC′, C′BA′ straightened out will be the sides of the second triangle $A_1B_1C_1$.

Suppose triangle ABC to be formed by an endless thread which marks out the perimeter. Take the mid points of BC, CA, AB, and stretch the thread between these points, and the second triangle is obtained.

The same process may be repeated on triangle $A_1B_1C_1$ and so on indefinitely. The limiting triangle which is thus obtained may be proved to be the equilateral triangle whose side is $\frac{1}{3}(a+b+c)$.

Can this process be extended backwards indefinitely far? To answer the question a solution must be found for the problem:

Given a triangle whose sides are a, b, c, construct the triangle whose sides are

$$-a+b+c, \quad a-b+c, \quad a+b-c.$$

FIGURE 24.

In triangle ABC inscribe the circle DEF;

then
$$AE = AF = \frac{-a+b+c}{2}$$

$$BF = BD = \frac{a-b+c}{2}$$

$$CD = CE = \frac{a+b-c}{2}.$$

Hence the triangle whose sides are equal to

$$AE + AF, \quad BF + BD, \quad CD + CE$$

will be the triangle sought.

Take, as before, the endless thread which marks out the perimeter of ABC at the points D, E, F and stretch it between these points.

Now this triangle is not always possible. For, in order that it may be possible, there must exist the inequality

$$a-b+c+a+b-c > -a+b+c, \text{ or } 3a > b+c.$$

Similarly $$3b > c + a, \text{ and } 3c > a + b.$$

By the addition of a, b, c these three inequalities may be transformed into $$2a > s, \ 2b > s, \ 2c > s.$$

But in every triangle the semiperimeter is greater than any one side. Hence the necessary and sufficient condition that the triangle antecedent to ABC may be possible is that each side of ABC must be greater than a quarter and less than a half of the perimeter.

The whole of (10) and a small part of (9) have been taken from a paper by Mr Édouard Collignon read at the Oran meeting (1888) of the *Association Française pour l'avancement des sciences*. See the Report of this meeting, Second Part, pp. 4-24. Mr Collignon's paper begins with a discussion of certain numerical series, and the results obtained are applied to the triangle, the quadrilateral, and to polygons of any number of sides.

§ 4. Excentres.

The internal bisector of any angle of a triangle and the external bisectors of the two other angles are concurrent.

The following demonstration is different from the usual one.

Figure 27.

Let AL be the internal bisector of $\angle$ A, and let the external bisector of $\angle$ B cut it at I_1.

Then
$$AI_1 : LI_1 = BA : BL$$
$$= CA : CL;$$

therefore the external bisector of $\angle$ C passes through I_1.

Hence also the internal bisector of $\angle$ B and the external bisectors of $\angle$ C and $\angle$ A are concurrent at I_2; the internal bisector of $\angle$ C and the external bisectors of $\angle$ A and $\angle$ B are concurrent at I_3.

Figure 28.

The points of concurrency, which will be denoted by I_1, I_2, I_3, are the centres of the circles escribed * to ABC.

These circles are often called the *excircles*,† and the centres of them the excentres.†

The radii of the excircles are denoted sometimes by r_1, r_2, r_3, sometimes by r_a, r_b, r_c.

(1) The points A, I, I_1; B, I, I_2; C, I, I_3 are collinear.

So are I_2, A, I_3; I_3, B, I_1; I_1, C, I_2.

These results expressed in words are:

The six bisectors of the interior and the exterior angles of a triangle meet three and three in four points which are the centres of the four circles touching the sides of the triangle. Or

The six straight lines joining two and two the centres of the four circles which touch the sides of a triangle pass each through one of the vertices of the triangle.

---

* This expression in its French form (*exinscrit*) was first used by Simon Lhuilier. See his *Élémens d'Analyse*, p. 198 (1809). If the term *escribed* was not introduced by T. S. Davies, currency at least was given to it by him. See *Ladies' Diary* for 1835, p. 50.

† See the note on p. 32.

(2) The points $\quad I, \quad I_1, \quad I_2, \quad I_3$
are the respective orthocentres of the triangles

$$I_1I_2I_3, \ II_3I_2, \ I_3II_1, \ I_2I_1I.$$

Attention should be called to the order of the subscripts in the naming of the triangles. See § 5, (2).

(3) When the diagram of a triangle with its incentres and excentres has to be drawn, instead of beginning with the triangle ABC and determining I, I_1, I_2, I_3 by the bisection of certain angles, it is easier to begin with the triangle $I_1I_2I_3$. The feet of the perpendiculars of $I_1I_2I_3$ will then be A, B, C, and I will be the point of intersection of the perpendiculars. The only instrument therefore which is necessary to determine these points is a draughtsman's square.

A circle may be escribed to a given triangle by a method exactly analogous to that on p. 39 for inscribing a circle.

FIGURE 24.

The only difference in the construction is that CQ is cut off equal to CB not in the same direction as CP, but in the opposite direction.

(4) The area of a triangle is equal to the rectangle under the excess of the semiperimeter above any side and the radius of the excircle which touches the other two sides produced.

This is expressed, $\quad \triangle = s_1r_1 = s_2r_2 = s_3r_3$
where $\quad s_1 = \frac{1}{2}(-a+b+c), \ s_2 = \frac{1}{2}(a-b+c), \ s_3 = \frac{1}{2}(a+b-c).$

If $\quad \frac{1}{2}(a+b+c)=s,$
then $\quad \frac{1}{2}(-a+b+c)=s-a, \ \frac{1}{2}(a-b+c)=s-b, \ \frac{1}{2}(a+b-c)=s-c.$

The expressions $s-a$, $s-b$, $s-c$ will be denoted by s_1, s_2, s_3, a notation introduced by Thomas Weddle, who in a letter to T. S. Davies, dated March 31st, 1842, and printed in the *Lady's and Gentleman's Diary* for 1843, p. 78, remarks that s, s_1, s_2, s_3 are the lengths of the segments of the sides made by the four circles of contact, and that the change from $s-a$ to s_1, etc., will be justified by observing how much more symmetrical many theorems appear under the new notation than the old.

(5) The distances from the vertices and from each other of some of the points of inscribed and escribed contact are given in the following expressions*:

* Compare the subscripts in the values of s, s_1, s_2, s_3 with the subscripts in § 4 (2).

FIGURE 28.

$$
\begin{aligned}
s &= AE_1 = AF_1 = BF_2 = BD_2 = CD_3 = CE_3\\
s_1 &= AE = AF = BF_3 = BD_3 = CD_2 = CE_2\\
s_2 &= AE_3 = AF_3 = BF = BD = CD_1 = CE_1\\
s_3 &= AE_2 = AF_2 = BF_1 = BD_1 = CD = CE
\end{aligned}
$$

$$
\begin{aligned}
a &= EE_1 = E_2E_3 = FF_1 = F_3F_2\\
b &= FF_2 = F_3F_1 = DD_2 = D_1D_3\\
c &= DD_3 = D_1D_2 = EE_3 = E_2E_1
\end{aligned}
$$

$$
\begin{aligned}
b+c &= D_2D_3 \quad & b \sim c &= DD_1\\
c+a &= E_3E_1 \quad & c \sim a &= EE_2\\
a+b &= F_1F_2 \quad & a \sim b &= FF_3
\end{aligned}
$$

(6)
$$
\begin{aligned}
a+b+c = 2s &= s+s_1+s_2+s_3\\
&= AE + AE_1 + AE_2 + AE_3 = \text{etc.}
\end{aligned}
$$

(7) Because $BD = s_2 = CD_1$

therefore D and D_1 are equidistant from the mid point of BC.

Because $BD_3 = s_1 = CD_2$

therefore D_2 and D_3 are equidistant from the mid point of BC.

Similarly for the E points and the F points.

(8)
$$
\begin{aligned}
&AD^2 + AD_1^2 + AD_2^2 + AD_3^2\\
&+ BE^2 + BE_1^2 + BE_2^2 + BE_3^2\\
&+ CF^2 + CF_1^2 + CF_2^2 + CF_3^2\\
&= 5(a^2+b^2+c^2).
\end{aligned}
$$

FIGURE 28.

Let A′ be the mid point of BC;

then A′ is the mid point of DD_1 and of D_2D_3.

Now since $D_2D_3 = b+c \qquad DD_1 = b \sim c,$

therefore $2A'D_2 = b+c \qquad 2A'D = b \sim c.$

But
$$
\begin{aligned}
AD^2 + AD_1^2 &= 2A'D^2 + 2A'A^2\\
AD_2^2 + AD_3^2 &= 2A'D_2^2 + 2A'A^2;
\end{aligned}
$$

therefore $\Sigma(AD^2) = 2A'D^2 + 2A'D_2^2 + 4A'A^2.$

Again $2AB^2 + 2AC^2 = 4A'B^2 + 4A'A^2$

$= BC^2 + 4A'A^2$;

therefore $2c^2 + 2b^2 - a^2 = 4A'A^2$.

Hence $\Sigma(AD^2) = \frac{1}{2}\{(b \sim c)^2 + (b+c)^2\} + 2c^2 + 2b^2 - a^2$

$= 3(b^2 + c^2) - a^2$.

Similarly $\Sigma(BE^2) = 3(c^2 + a^2) - b^2$

and $\Sigma(CF^2) = 3(a^2 + b^2) - c^2$.

Consequently the sum of the squares on the twelve specified lines

$= 5(a^2 + b^2 + c^2)$.

(8) is stated by W. H. Levy of Shalbourne in the ***Lady's and Gentleman's Diary*** for 1852, p. 71, and proved in 1853, pp. 52-3.

(9) The angles of triangle DEF expressed in terms of A, B, C are

$$\angle D = \tfrac{1}{2}(B + C) = 90° - \tfrac{1}{2}A$$
$$\angle E = \tfrac{1}{2}(C + A) = 90° - \tfrac{1}{2}B$$
$$\angle F = \tfrac{1}{2}(A + B) = 90° - \tfrac{1}{2}C.$$

Hence whatever be the size of the angles A, B, C the triangle DEF is always acute-angled.*

(10) *If ABC be a triangle, DEF the triangle formed by joining the inscribed points of contact of ABC; $D_1E_1F_1$ the triangle formed by joining the inscribed points of contact of DEF; $D_2E_2F_2$ the triangle formed by joining the inscribed points of contact of $D_1E_1F_1$; and this process of construction be continued, the successive triangles will approximate to an equilateral triangle.*†

Suppose $\angle A$ greater than $\angle B$, and $\angle B$ greater than $\angle C$.

Then $D = \frac{1}{2}(B + C)$, $E = \frac{1}{2}(C + A)$, $F = \frac{1}{2}(A + B)$;

therefore $E - D = \frac{1}{2}(A - B)$, $F - E = \frac{1}{2}(B - C)$, $F - D = \frac{1}{2}(A - C)$.

Now $D_1 = \frac{1}{2}(E + F)$, $E_1 = \frac{1}{2}(F + D)$, $F_1 = \frac{1}{2}(D + E)$;

therefore $D_1 - E_1 = \frac{1}{2}(E - D)$, $E_1 - F_1 = \frac{1}{2}(F - E)$, $D_1 - F_1 = \frac{1}{2}(F - D)$,

$= \frac{1}{4}(A - B)$, $= \frac{1}{4}(B - C)$, $= \frac{1}{4}(A - C)$;

and so on.

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* Feuerbach, *Eigenschaften...des...Dreiecks*, § 66 (1822).

† Todhunter's ***Plane*** *Trigonometry*, Chapter XVI. Example 16 (1859).

Therefore the differences between the angles of the successive triangles become always a smaller and smaller fraction of the differences between the angles of the fundamental triangle; and hence the successive triangles approximate to an equilateral triangle.

The properties (11)–(14), (16)–(18) have reference to Figure 28.

(11) EF, E_1F_1 are parallel to I_2I_3
and EF_1, E_1F intersect on AI; and so on.

(12) E_2F_2, E_3F_3 are parallel to AI
and E_2F_3, E_3F_2 intersect on I_2I_3; and so on.

(13) The angles of triangles $D_1E_1F_1$, $D_2E_2F_2$, $D_3E_3F_3$ expressed in terms of A, B, C are

$$\begin{array}{lll}
\angle D_1 = 90^\circ + \tfrac{1}{2}A, & \angle E_1 = \tfrac{1}{2}B, & \angle F_1 = \tfrac{1}{2}C \\
\angle D_2 = \tfrac{1}{2}A, & \angle E_2 = 90^\circ + \tfrac{1}{2}B, & \angle F_2 = \tfrac{1}{2}C \\
\angle D_3 = \tfrac{1}{2}A, & \angle E_3 = \tfrac{1}{2}B, & \angle F_3 = 90^\circ + \tfrac{1}{2}C.
\end{array}$$

Hence whatever be the size of the angles A, B, C these three triangles are always obtuse-angled.*

(14) The angles of triangle $I_1I_2I_3$ expressed in terms of A, B, C are

$$\begin{aligned}
\angle I_1 &= \tfrac{1}{2}(B+C) = 90^\circ - \tfrac{1}{2}A \\
\angle I_2 &= \tfrac{1}{2}(C+A) = 90^\circ - \tfrac{1}{2}B \\
\angle I_3 &= \tfrac{1}{2}(A+B) = 90^\circ - \tfrac{1}{2}C
\end{aligned}$$

Hence whatever be the size of the angles A, B, C the triangle $I_1I_2I_3$ is always acute-angled.

(15) If ABC be a triangle, $A_1B_1C_1$ the triangle formed by joining the excentres of ABC; $A_2B_2C_2$ the triangle formed by joining the excentres of $A_1B_1C_1$; and this process of construction be continued, the successive triangles will approximate to an equilateral triangle.†

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* Feuerbach, *Eigenschaften...des...Dreiecks*, § 66 (1822).

† Mr R. Tucker in *Mathematical Questions from the Educational Times*, XV. 103-4 (1871).

(16) Triangles DEF, $I_1I_2I_3$ are similar and similarly situated,* and their homothetic centre is the point of concurrency of the triad

$$I_1D,\ I_2E,\ I_3F.$$

(17) Triangles $D_1E_1F_1$, II_3I_2 are similar and similarly situated; so are $D_2E_2F_2$, I_3II_1; and $D_3E_3F_3$, I_2I_1I; and their homothetic centres are the points of concurrency of the triads

$$ID_1,\ I_3E_1,\ I_2F_1;\ \text{and so on.}$$

(18) *The quadrilaterals AFIE, BDIF, CEID are such that circles may be inscribed in them.*

For a circle may be inscribed in a quadrilateral when the sum of one pair of opposite sides is equal to the sum of the other pair.

Now $\qquad AF = AE$ and $IE = IF$.

(19) *If the radii of the circles inscribed in the quadrilaterals*† *AFIE, BDIF, CEID* be denoted by $\rho_1, \rho_2, \rho_3,$

$$\left(\frac{1}{\rho_2}-\frac{1}{r}\right)\left(\frac{1}{\rho_3}-\frac{1}{r}\right)+\left(\frac{1}{\rho_3}-\frac{1}{r}\right)\left(\frac{1}{\rho_1}-\frac{1}{r}\right)+\left(\frac{1}{\rho_1}-\frac{1}{r}\right)\left(\frac{1}{\rho_2}-\frac{1}{r}\right)=\frac{1}{r^2}.$$

FIGURE 29.

Bisect $\angle$ AFI by FM meeting AI at M; and draw MN perpendicular to AF.

Then M is the centre of the circle inscribed in AFIE, MN is the radius of it, and MN = FN.

From the similar triangles AFI, ANM

$$AF : IF = AN : MN$$

that is, $\qquad s_1 : r = s_1 - \rho_1 : \rho_1$;

therefore $\qquad s_1 : r = \rho_1 : r - \rho_1$;

therefore $$s_1 = \frac{r\rho_1}{r-\rho_1};$$

therefore $$\frac{1}{s_1} = \frac{1}{\rho_1} - \frac{1}{r}.$$

Similarly $$\frac{1}{s_2} = \frac{1}{\rho_2} - \frac{1}{r},\quad \frac{1}{s_3} = \frac{1}{\rho_3} - \frac{1}{r}.$$

Hence given expression $$= \frac{1}{s_2s_3} + \frac{1}{s_3s_1} + \frac{1}{s_1s_2} = \frac{1}{r^2}.$$

---

* Feuerbach, *Eigenschaften ... des ... Dreiecks*, § 61 (1822).

† *The Museum*, III. 269-70 and 342 (1866).

(20) $$\frac{\rho_1}{r-\rho_1}+\frac{\rho_2}{r-\rho_2}+\frac{\rho_3}{r-\rho_3} = \frac{\rho_1}{r-\rho_1}\cdot\frac{\rho_2}{r-\rho_2}\cdot\frac{\rho_3}{r-\rho_3}.$$

For $$\frac{r\rho_1}{r-\rho_1}=s_1;$$

therefore $$\frac{\rho_1}{r-\rho_1}=\frac{s_1}{r}.$$

Similarly $$\frac{\rho_2}{r-\rho_2}=\frac{s_2}{r},\quad \frac{\rho_3}{r-\rho_3}=\frac{s_3}{r};$$

therefore $$\frac{\rho_1}{r-\rho_1}+\frac{\rho_2}{r-\rho_2}+\frac{\rho_3}{r-\rho_3}=\frac{s_1+s_2+s_3}{r}=\frac{s}{r}$$

and $$\frac{\rho_1}{r-\rho_1}\cdot\frac{\rho_2}{r-\rho_2},\frac{\rho_3}{r-\rho_3}=\frac{s_1s_2s_3}{r^3}=\frac{s}{r}.$$

(21) *The quadrilaterals $AF_1I_1E_1$, $BD_1I_1F_1$, $CE_1I_1D_1$ are such that circles may be inscribed in them.*

FIGURE 28.

For $$AF_1 = AE_1 \text{ and } I_1E_1 = I_1F_1.$$

Similarly, circles may be inscribed in the quadrilaterals

$$AF_2I_2E_2,\ BD_2I_2F_2,\ CE_2I_2D_2$$
$$AF_3I_3E_3,\ BD_3I_3F_3,\ CE_3I_3D_3.$$

(22) If the radii of the circles inscribed in the first three of these quadrilaterals be denoted by ρ_1', ρ_2', ρ_3' then

$$\left(\frac{1}{\rho_2'}-\frac{1}{r_1}\right)\left(\frac{1}{\rho_3'}-\frac{1}{r_1}\right)-\left(\frac{1}{\rho_3'}-\frac{1}{r_1}\right)\left(\frac{1}{\rho_1'}-\frac{1}{r_1}\right)-\left(\frac{1}{\rho_1'}-\frac{1}{r_1}\right)\left(\frac{1}{\rho_2'}-\frac{1}{r_1}\right)=\frac{1}{r_1^2};$$

and similarly for the others.

(23) $$\frac{\rho_1'}{r_1-\rho_1'}-\frac{\rho_2'}{r_1-\rho_2'}-\frac{\rho_3'}{r_1-\rho_3'}=\frac{\rho_1'}{r_1-\rho_1'}\cdot\frac{\rho_2'}{r_1-\rho_2'}\cdot\frac{\rho_3'}{r_1-\rho_3'}$$

For $$s=\frac{r_1\rho_1'}{r_1-\rho_1'},\quad s_3=\frac{r_1\rho_2'}{r_1-\rho_2'},\quad s_2=\frac{r_1\rho_3'}{r_1-\rho_3'}.$$

(24) The following relation* exists between the radii of the circles inscribed in the quadrilaterals

$$AF_1I_1E_1,\ BD_2I_2F_2,\ CE_3I_3D_3.$$

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* Mr R. E. Anderson in *Proceedings of the Edinburgh Mathematical Society*, X. 79 (1892).

If these radii be denoted by v_1, v_2, v_3

$$\frac{1}{v_1}+\frac{1}{v_2}+\frac{1}{v_3}=\frac{3}{s}+\frac{1}{r}.$$

(25) *If the exterior angles of ABC be bisected by straight lines which meet the circumcircle at U′, V′, W′, the sides of U′V′W′ are also perpendicular to AI, BI, CI, and U′V′W′ is congruent and symmetrically situated to UVW with respect to the circumcentre of ABC.*

FIGURE 25.

For $\angle$ UAU′ is right;
therefore UU′ is a diameter of the circumcircle,
and U′ is symmetrical to U with respect to O.

(26) If from the six points U, V, W, U′, V′, W′ all the UVW triangles be formed, it will be found that there are four pairs

UVW, UV′W′, U′VW′, U′V′W
U′V′W′, U′VW, UV′W, UVW′.

These pairs of triangles are congruent and symmetrically situated with respect to O, and their sides are either perpendicular or parallel to AI, BI, CI.

(27) The angles of these triangles can be expressed in terms of A, B, C.

Take, for example, triangle U′VW from the second pair.

$$\begin{aligned} \text{WU'V} &= 180^\circ - \text{WUV} \\ &= 180^\circ - \tfrac{1}{2}(\text{B}+\text{C}) = 90^\circ + \tfrac{1}{2}\text{A}. \end{aligned}$$

Since AU′, VW are both perpendicular to AI,

therefore arc U′W = arc AV
therefore $\angle$ U′VW = $\angle$ ABV = $\frac{1}{2}$B,
and $\angle$ U′WV = $\angle$ ACW = $\frac{1}{2}$C.

The angles of the third and fourth pairs of triangles are respectively equal to

$$\begin{matrix} \tfrac{1}{2}\text{A}, & 90^\circ+\tfrac{1}{2}\text{B}, & \tfrac{1}{2}\text{C} \\ \tfrac{1}{2}\text{A}, & \tfrac{1}{2}\text{B}, & 90^\circ+\tfrac{1}{2}\text{C}. \end{matrix}$$

Hence whatever be the size of the angles A, B, C the second, third, and fourth pairs of triangles are always obtuse-angled.

Compare § 4, (13).

(28) The orthocentres of the first quartet of triangles are

$$I, I_1, I_2, I_3$$

and the orthocentres of the second quartet are the points symmetrical to I, I_1, I_2, I_3 with respect to O. What these points are will be seen later on.

(29)
$$\begin{aligned} UV'W' &: ABC = R : 2r_1 \\ U'VW' &: ABC = R : 2r_2 \\ U'V'W &: ABC = R : 2r_3. \end{aligned}$$

(30) If I be the incentre of ABC and about the triangles IBC, ICA, IAB circles be circumscribed, these circles will pass respectively through I_1, I_2, I_3, and their centres will be U, V, W, the points where AI, BI, CI meet the circumcircle.

FIGURE 28.

(31) If circles be circumscribed about the triangles I_1BC, I_1CA, I_1AB, these circles will pass respectively through I, I_3, I_2.

Similarly for the circumcircles of I_2BC, etc.

(32) From (30) and (31) there would seem to be twelve circles. They reduce to six, and their diameters are the six lines

$$II_1,\ II_2,\ II_3,\ I_2I_3,\ I_3I_1,\ I_1I_2.$$

(33) *In a triangle ABC, if on AB and AC as diameters circles be described and a diameter of the first circle be drawn parallel to AC, and a diameter of the second parallel to AB, one pair of extremities of these diameters will lie on the internal bisector of angle A and the other pair on the external bisector.**

FIGURE 30.

Let M and N be the centres of the circles described on AB and AC, and let EF be the diameter parallel to AC, and GH the diameter parallel to AB. Join AF.

Then $\angle MAF = \angle MFA = \angle NAF$;

therefore AF is the internal bisector of $\angle A$.

Similarly AH is the internal bisector of $\angle A$.

Join AE, and produce CA to C′.

* Mr Brocot in the *Journal de Mathématiques Élémentaires*, I. 383 (1877), II. 128 (1878).

Then $\angle MAE = \angle MEA = \angle C'AE$;
therefore AE is the external bisector of $\angle A$.
Similarly AG is the external bisector of $\angle A$.

(34) *If M, N be the feet of the bisectors of angles B and C of triangle ABC, the distance of any point P in MN from BC is equal to the sum* of its distances from CA and AB.*

FIGURE 31.

Draw MK, MK′ perpendicular to BC, AB
NL, NL′ „ „ BC, CA;
join ML meeting PR in H.

Then $PS : NL' = PM : NM$
$= PH : NL.$
But $NL' = NL$;
therefore $PS = PH$.
Again $PT : MK' = PN : MN$
$= RL : KL$
$= HR : MK.$
But $MK' = MK$;
therefore $PT = HR$;
therefore $PR = PS + PT.$

If P be situated on MN produced either way, then PR is equal to the difference between PS and PT.

The theorem may be extended to the bisectors of the exterior angles of ABC, and thus enunciated:

If M, N be the feet of the internal or external bisectors of the angles B and C, the distance of any point P in MN from BC is equal to the algebraic sum of its distances from CA and AB.

(35) If through I, I_1, I_2, I_3 parallels be drawn to BC meeting AC, AB in P, Q; P_1, Q_1; P_2, Q_2; P_3, Q_3

then $PQ = BQ + CP$, $P_1Q_1 = BQ_1 + CP_1$
$P_2Q_2 = BQ_2 - CP_2$, $P_3Q_3 = BQ_3 - CP_3$.

* Mr E. Cesaro in *Nouvelle Correspondance Mathématique*, V. 224 (1879); proof and extension of the property to the external bisectors by Mr Cauret on pp. 334-5. The proof in the text is taken from Vuibert's *Journal* IX. 72 (1885). Mr Cesaro gives the corresponding property for the tetrahedron.

(36) *If AI meet the incircle at U, and at U a tangent be drawn meeting BI and CI at P and Q, then**

$$BP = CQ = BI + CI.$$

FIGURE 32.

Let PQ meet CA, AB at S, T.

Then $\angle QTB = \angle ATS$
$= \frac{1}{2}(B + C)$;

and $\angle QIB = \angle IBC + \angle ICB$
$= \frac{1}{2}(B + C)$;

therefore the points B, I, T, Q are concyclic,

and $\angle IQU = \angle IBF$.

Hence the right-angled triangles IUQ, IFB, which have IU equal to IF, are congruent;

therefore $QI = BI$.

Similarly $PI = CI$;

therefore $BP = CQ = BI + CI$.

(37) $BT = QS$ and $CS = PT$.

For $BT = BF + TF = QU + TU = QU + SU$.

(38) If AI_1 meet the first excircle at U_1, and at U_1 a tangent be drawn meeting BI_1 and CI_1 at P_1 and Q_1, then

$$BP_1 = C\,Q_1 = BI_1 + CI_1.$$

(39) $PQ = P_1Q_1 = BC$.

(40) $BT_1 = Q_1S_1$ and $CS_1 = P_1T_1$.

(41) *If ID, the radius of the incircle, meet EF at P, then*† *P lies on the median through A.*

FIGURE 33.

Through P draw B_1C_1 parallel to BC, and join IB_1, IC_1.

Then $\angle IPB_1$ and $\angle IFB_1$ are right;

therefore the points I, P, F, B_1 are concyclic;

therefore $\angle IB_1P = \angle IFP$.

---

* This and (37) are given by William Wallace in Leybourn's *Mathematical Repository*, old series, II. 187 (1801).

† John Johnson, of Birmingham, in Leybourn's *Mathematical Repository*, old series, II. 376 (1801).

Mr E. M. Langley in the *Sixteenth Report of the Association for the Improvement of Geometrical Teaching*, pp. 35-6 (1890), gives another demonstration by means of Brianchon's theorem:

Similarly $\angle IC_1P = \angle IE\,P$;

therefore $\angle IB_1P = \angle IC_1P$;

therefore $B_1P = C_1P$.

Hence also if I_1D_1, the radius of the first excircle, meet E_1F_1 at P_1, then P_1 lies on the median through A.

(42) *If from any point P situated on the interior or exterior bisector of the angle A of triangle ABC perpendiculars PD, PE, PF be drawn to BC, CA, AB, the point Q where PD intersects EF will lie on the median * from A.*

FIGURE 34.

Triangles FPQ, ABL are similar, since their sides are mutually perpendicular;

therefore $FQ : FP = AL : AB$.

Similarly $EQ : EP = AL : AC$;

therefore $AB \cdot FQ = AC \cdot EQ$.

Now FQ and EQ are proportional to the distances of Q from AB and AC;

therefore $ABQ = ACQ$.

But these triangles have the same base AQ;

therefore their corresponding altitudes are equal;

and hence it is easily deduced that AQ passes through the mid point of BC.

The following is another demonstration:

When the point P moves on the bisector, the point Q describes a straight line passing through A.

Place the point P at the intersection of the bisector with the circumcircle of ABC;

then the projections of P on the sides of ABC are collinear, by Wallace's theorem;

and one of these projections is the mid point of BC.

* Mr E. Cesaro in *Mathesis* I. 79 (1881). The two demonstrations are from the same volume, pp. 117-8.

(43) *The problem of finding the incentre or the excentres of a triangle is a particular case of the problem to find a point such that straight lines drawn from it to the sides shall make equal angles with the sides and shall be to each other in given ratios.**

FIGURE 35.

Let the straight lines drawn from the point to BC, CA, AB be in the ratios $d : e : f$.

Make a parallelogram having B for one of its angles, and having the sides along BA, BC in the ratio $d : f$;
let BM be one of its diagonals.

Make a parallelogram having C for one of its angles, and having the sides along CA, CB in the ratio $d : e$;
let CN be one of its diagonals.
Then BM, CN will meet at I the required point.

From I draw ID, IE, IF making with BC, CA, AB angles equal to the given angle.

The proof will offer no great difficulty if from M perpendiculars be drawn to BA, BC, from N to CA, CB, and from I to the three sides.

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* Mauduit's *Leçons de Géométrie*, pp. 239-242 (1790).

§ 5. Orthocentre.

*The perpendiculars to the sides of a triangle from the opposite vertices are concurrent.**

One of the earliest demonstrations occurs in Pierre Herigone's *Cursus Mathematicus*, I. 318 (1634). Three cases are considered, when the triangle is right-angled, acute-angled, obtuse-angled.

From the various proofs that have been published, the following are selected.

First Demonstration.†

Figure 36.

Let AX, BY which are perpendicular to BC, CA meet at H, and let CH be joined and produced to meet AB at Z.

Join XY.

Because $\angle$ AXC and $\angle$ BYC are right,

therefore C, X, H, Y are concyclic, as well as A, Y, X, B;

therefore $\angle ACZ = \angle AXY$,

$= \angle ABY$.

Now $\angle$ ZAY is common to triangles ACZ, ABY;

therefore $\angle AZC = \angle AYB$,

= a right angle.

Second Demonstration.‡

Figure 37.

Let AX, BY, CZ be the three perpendiculars from A, B, C on BC, CA, AB.

Through A, B, C draw B_1C_1, C_1A_1, A_1B_1 respectively parallel to BC, CA, AB.

---

* This theorem occurs without proof in the fifth of the *Lemmas* ascribed to Archimedes, and also in Pappus's *Mathematical Collection*, VII. 62. In Commandino's editions of Pappus, which were published after his death, the proof supplied is erroneous. The mistake has been noticed by several mathematical writers.

† Robert Simson's *Opera Quaedam Reliqua*, p. 171 (1776).

‡ This mode of proof is given by F. J. Servois in his *Solutions peu connues de différens problèmes de Géométrie-pratique*, p. 15 (1804). It was also given by Gauss, and will be found in Schumacher's translation into German of Carnot's *Géométrie de Position*, II. 363 (1810).

Then $ABCB_1$, $ACBC_1$ are parallelograms,
and A is the mid point of B_1C_1.
Hence also B and C are the mid points of C_1A_1 and A_1B_1.
But AX, BY, CZ are respectively perpendicular to BC, CA, AB;
therefore they must be respectively perpendicular to B_1C_1, C_1A_1, A_1B_1.
If therefore it be assumed as true that the perpendiculars to the sides of a triangle from the mid points of the sides are concurrent,

AX, BY, CZ are concurrent.

THIRD DEMONSTRATION.*

FIGURE 38.

Let AX, BY, CZ be the three perpendiculars from A, B, C on BC, CA, AB.

Join YZ, ZX, XY.

Since the points A, Z, X, C are concyclic,

therefore $\angle BXZ = \angle BAC$.

Since the points A, Y, X, B are concyclic,

therefore $\angle CXY = \angle BAC$;

therefore $\angle BXZ = \angle CXY$.

Now $\angle BXA = \angle CXA$;

therefore AX bisects $\angle ZXY$.

Hence BY „ $\angle XYZ$,

and CZ „ $\angle YZX$.

If therefore it be assumed as true that the internal angular bisectors of a triangle are concurrent

AX, BY, CZ are concurrent.

FOURTH DEMONSTRATION.†

"If three straight lines drawn through the vertices of a triangle are concurrent, their isogonals with respect to the angles of the triangle are also concurrent."

This theorem, which is due to Steiner,‡ taken along with the property, which is established in the proof of Brahmegupta's theorem, namely,

* Mr Bernh. Möllmann in Grunert's *Archiv*, XVII., 376 (1851).

† Dr James Booth's *New Geometrical Methods*, II. 260-1 (1877).

‡ Gergonne's *Annales*, XIX. 37-64 (1828), or Steiner's *Gesammelte Werke*, I. 193 (1881).

"The perpendicular from any vertex of a triangle to the opposite side and the diameter of the circumcircle drawn from that vertex are isogonal with respect to the vertical angle" furnishes a ready proof. For the diameters of the circumcircle are concurrent.

The point H, where AX, BY, CZ are concurrent, is now generally called the orthocentre* of ABC; and the triangle XYZ is called sometimes the orthic,† sometimes the orthocentric,‡ and sometimes the pedal, triangle.

It may be noted that H is the initial letter in English, French, and German of the names for AX, BY, CZ (*Heights*, *Hauteurs*, *Höhen*).

(1) If in Fig. 37 ABC be considered the fundamental triangle, $A_1B_1C_1$ is anticomplementary to it, and hence the orthocentre of any triangle is the circumcentre of the anticomplementary triangle.

If however $A_1B_1C_1$ be considered the fundamental triangle, ABC is complementary to it, and hence the circumcentre of any triangle is the orthocentre of the complementary triangle.

(2) The four points A, B, C, H, taken three by three form four triangles ABC, HCB, CHA, BAH; of these four triangles the fourth points H, A, B, C are the respective orthocentres, and in all the four cases the orthic triangle is XYZ. "The figure is therefore a system of four points joined two and two by straight lines such that each of them passing through two of these points cuts perpendicularly that which passes through the two others." §

In naming the four triangles the order of the letters is such that X is the foot of the perpendicular from the vertex first named, Y the foot of that from the second named vertex, and Z the foot of that from the third. This is a matter of much more importance than appears at first sight.

It may be convenient to call a set of four points such as A, B, C, H an *orthic tetrastigm*.

* This useful expression was suggested by Dr Ferrers and Dr W. H. Besant in 1866-7. It is introduced in Dr Besant's *Conic Sections*, § 138 (1869).

† Mr Emile Vigarié in *Mathesis*, VII. 61 (1887).

‡ Dr James Booth in his *New Geometrical Methods*, II. 261 (1877).

§ Carnot, *Corrélation des Figures de Géométrie*, § 143 (1801).

(3) The angles of the triangles HCB, CHA, BAH expressed in terms of A, B, C are

$$\begin{array}{lll} \angle\, BHC = 180^\circ - A, & \angle\, HCB = \ \ 90^\circ - B, & \angle\, CBH = \ \ 90^\circ - C \\ \angle\, ACH = \ \ 90^\circ - A, & \angle\, CHA = 180^\circ - B, & \angle\, HAC = \ \ 90^\circ - C \\ \angle\, HBA = \ \ 90^\circ - A, & \angle\, BAH = \ \ 90^\circ - B, & \angle\, AHB = 180^\circ - C. \end{array}$$

(4) *The fundamental triangle is inversely similar to the triangles "cut off" from it by the sides of the orthic triangle.*

FIGURE 38.

If ABC be the fundamental triangle, H is its orthocentre, XYZ its orthic triangle, and the triangles cut off from ABC and similar to it are AYZ, XBZ, XYC.

If HCB be taken as the fundamental triangle, A is its orthocentre, XYZ its orthic triangle, and the triangles "cut off" from HCB and similar to it are HYZ, XCZ, XYB.

Similarly for CHA and triangles CYZ, XHZ, XYA
and for BAH „ „ BYZ, XAZ, XYH.

(5) ABC is the orthic triangle not only of $I_1I_2I_3$, but also of II_3I_2, I_3II_1, I_2I_1I.

FIGURE 28.

Hence the sides of ABC "cut off" from these four triangles four triads of triangles which are respectively similar to them. They are

$$\begin{array}{lll} \text{To } I_1I_2I_3\,; & I_1BC, & AI_2C, & ABI_3 \\ \text{„ } I\,I_3I_2\,; & I\,BC, & AI_3C, & ABI_2 \\ \text{„ } I_3I\,I_1\,; & I_3BC, & AI\,C, & ABI_1 \\ \text{„ } I_2I_1I\,; & I_2BC, & AI_1C, & ABI\,. \end{array}$$

(6) The following triads of lines form by their intersections four triangles which are similar and oppositely situated to the four triangles of the orthic tetrastigm $II_1I_2I_3$.

FIGURE 28.

Lines.	Triangles.
E_1F_1, F_2D_2, D_3E_3	$I_1I_2I_3$
$E\,F$, F_3D_3. D_2E_2	$I\,I_3I_2$
E_3F_3, $F\,D$, D_1E_1	$I_3I\,I_1$
E_2F_2, F_1D_1, $D\,E$	I_2I_1I

Compare the subscripts in the naming of the lines with the subscripts in the naming of the triangles.

(7) *The sides of the orthic triangle are respectively antiparallel* to those of the fundamental triangle with respect to the angles of the fundamental triangle.*

FIGURE 38.

If ABC be taken as the fundamental triangle,

YZ	is	antiparallel	to BC	with	respect	to	$\angle$ CAB,
ZX	„	„	„ CA	„	„	„	$\angle$ ABC
XY	„	„	„ AB	„	„	„	$\angle$ BCA.

If HCB be taken as the fundamental triangle,

YZ	is	antiparallel	to CB	with	respect	to	$\angle$ BHC
ZX	„	„	„ BH	„	„	„	$\angle$ HCB
XY	„	„	„ HC	„	„	„	$\angle$ CBH.

Similarly for the triangles CHA, BAH.

(8) The angles of triangle XYZ expressed in terms of A, B, C are:

$$\angle X = 180° - 2A = -A + B + C$$
$$\angle Y = 180° - 2B = A - B + C$$
$$\angle Z = 180° - 2C = A + B - C.$$

(9) If ABC, XYZ, $X_1Y_1Z_1$, $X_2Y_2Z_2$ ...... be a series of triangles such that each is the orthic triangle of the preceding, the following tabular statements of their angles may be given.*

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* Carnot's *Géométrie de Position*, § 151 (1803). The term antiparallel was first used by Antoine Arnauld. See *Nouveaux Éléments de Géométrie*, par Messrs de Port-Royal, p. 212, or livre onzième (1667). Further information regarding the use of the word will be found in two letters from Mr E. M. Langley to *Nature*, XL., 460-1 (1889), and XLI., 104-5 (1889).

* These are taken from an article by Mr H. Brocard in the *Nouvelle Correspondance Mathématique*, VI. 145-151 (1880).

Triangles.	Angles.		
	A	B	C
A B C	A	B	C
X Y Z	$-A+B+C$	$A-B+C$	$A+B-C$
$X_1Y_1Z_1$	$3A-B-C$	$-A+3B-C$	$-A-B+3C$
$X_2Y_2Z_2$	$-5A+3B+3C$	$3A-5B+3C$	$3A+3B-5C$
$X_3Y_3Z_3$	$11A-5B-5C$	$-5A+11B-5C$	$-5A-5B+11C$
......	......	......	......

Consider the coefficients (all taken with the positive sign) of the angle A in the first column of angles. They form the series

$$\begin{array}{cccccccc} u_0 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 \\ 1 & 3 & 5 & 11 & 21 & 43 & 85 & 171 \end{array}$$

where the law of recurrence is

$$u_{n+1} = u_n + 2u_{n-1}$$

with the initial conditions $u_0 = 1$, $u_1 = 3$.

Triangles.	Angles.		
A B C	A	B	C
X Y Z	$\pi - 2A$	$\pi - 2B$	$\pi - 2C$
$X_1Y_1Z_1$	$4A - \pi$	$4B - \pi$	$4C - \pi$
$X_2Y_2Z_2$	$3\pi - 8A$	$3\pi - 8B$	$3\pi - 8C$
$X_3Y_3Z_3$	$16A - 5\pi$	$16B - 5\pi$	$16C - 5\pi$
......	......	......	......

In these expressions the coefficient of A, B, or C is a power of 2, and the coefficient of π is one term of the series $u_0\ u_1\ u_2\ u_3 \ldots$

The angle $X_{2n} = u_{2n-1}\pi - 2^{2n+1}A$;

" " $X_{2n-1} = 2^{2n}\ A - u_{2n-2}\pi$.

(10) *The orthocentre and the vertices of the fundamental triangle are the incentre and the excentres of the orthic triangle.**

* Feuerbach, *Eigenschaften...des...Dreiecks*, § 24 (1822).

FIGURE 38.

In Möllmann's demonstration of the concurrency of the perpendiculars, it was shown that, if ABC be taken as the fundamental triangle, H is the incentre of XYZ.

Now since BC, CA, AB are respectively perpendicular to AX, BY, CZ, therefore BC, CA, AB are the bisectors of the external angles of XYZ;
therefore A, B, C are the excentres of XYZ.

If HCB be taken as the fundamental triangle, its vertices, B, C and its orthocentre A are the excentres of XYZ, and the vertex H is the incentre.

Similarly for the triangles CHA, BAH.

(11) *If from the mid points of YZ, ZX, XY perpendiculars be drawn to BC, CA, AB, these perpendiculars are concurrent.**

If X′, Y′, Z′ be the mid points,
then triangle X′Y′Z′ is similar and oppositely situated to XYZ;
therefore the respective perpendiculars are the bisectors of the angles of X′Y′Z′, and consequently concurrent at the incentre of X′Y′Z′.

(12) The perpendiculars from X′, Y′, Z′ respectively to

$$\left.\begin{array}{l}\text{CB, BH, HC}\\ \text{HA, AC, CH}\\ \text{AH, HB, BA}\end{array}\right\}\text{ are concurrent at the }\left\{\begin{array}{llll}\text{first} & \text{excentre} & \text{of} & \text{X}'\text{Y}'\text{Z}'\\ \text{second} & \text{,,} & \text{,,} & \text{,,}\\ \text{third} & \text{,,} & \text{,,} & \text{,,}\end{array}\right.$$

These four points, the incentre and the excentres of triangle X′Y′Z′, will be considered again, in connection with the Taylor circles.

(13) *If the perpendiculars of a triangle meet the circumcircle again in R, S, T, then R, S, T are the images of the orthocentre in the sides.*

FIGURE 39.

Let ABC be the triangle, H its orthocentre.

Join BR.

Then $\angle \text{CBY} = \angle \text{CAX} = \angle \text{CBR}$;

therefore the right-angled triangles BXH, BXR are congruent,

and $\text{HX} = \text{RX}$.

Similarly $\text{HY} = \text{SY}$ and $\text{HZ} = \text{TZ}$.

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* Édouard Lucas in *Nouvelle Correspondance Mathématique*, II. 95, 218 (1876).

If HCB be taken as the triangle instead of ABC, then A is its orthocentre, HX, CY, BZ its perpendiculars. Let a circle be circumscribed about HCB, and let the perpendiculars meet it again at R_1, S_1, T_1.

FIGURE 40.

Then it may be shown as before that

$$AX = R_1X, \quad AY = S_1Y, \quad AZ = T_1Z.$$

Similarly for the triangles CHA, BAH.

(14) *The triangles RST, XYZ are similar and similarly situated; H is their homothetic centre, and their ratio of similitude is* 2 : 1.

FIGURE 40.

Since X, Y, Z are the mid points of HR, HS, HT, therefore the sides of XYZ are respectively parallel to those of RST, and equal to the halves of them.

In like manner the triangles $R_1S_1T_1$, XYZ are similar and similarly situated; A is their homothetic centre, and their ratio of similitude is 2 : 1.

(15) H is the incentre of R S T,
A „ „ first excentre „ $R_1S_1T_1$.

Similarly for B and C.

(16) *The circumcircle of ABC is equal* to the circumcircles of HCB, CHA, BAH.*

FIGURE 39.

For triangle HCB is congruent to RCB;
and the circumcircle of RCB is the circumcircle of ABC.

(17) *If O_a, O_b, O_c be the centres of the circumcircles of HCB, CHA, BAH, then triangle $O_aO_bO_c$ is congruent, and oppositely situated, to ABC.*

FIGURE 41.

For O_bO_c, O_cO_a, O_aO_b are perpendicular to HA, HB, HC
and BC, CA, AB „ „ „ „ „ „ .

* Carnot's *Corrélation des Figures de Géométrie*, § 146 (1801), or *Géométrie de Position*, § 130 (1803).

(18) *H is the circumcentre of $O_aO_bO_c$,*
O „ „ orthocentre „ „ .

Since the circles O_b, O_c are equal,
therefore O_bO_c bisects, and is bisected by, their common chord HA perpendicularly;

therefore $HO_b = HO_c$.
Similarly $HO_c = HO_a$.

Again, since the circles O, O_a are equal,
therefore OO_a bisects, and is bisected by, their common chord BC perpendicularly;

therefore O_aO is perpendicular to O_bO_c.
Similarly O_bO „ „ „ O_cO_a.

(19) The points O_a, O_b, O_c, O form an orthic tetrastigm, congruent and oppositely situated to the orthic tetrastigm A, B, C, H.

(20) If through A any straight line be drawn meeting the circles O_b, O_c in M, N, then MC, NB will meet on the circumference of O_a.

(21) If any point L be taken on the circumference of O_a, and LC, LB meet the circumferences of O_b, O_c again in M, N, then the points M, A, N are collinear, and triangle LMN is directly similar to ABC.

(22) Of all the triangles such as LMN whose sides pass through A, B, C, and whose vertices are situated on the circles O_a, O_b, O_c, that triangle $A_1B_1C_1$ is a maximum whose sides are perpendicular to AH, BH, CH.

Compare § 2, (15) – (19).

(23) *Triangle $A_1B_1C_1$ is the anticomplementary triangle of ABC; it has H for its circumcentre, and its circumcircle touches the circles O_a, O_b, O_c at the points A_1, B_1, C_1.*

For A, B, C are the mid points of B_1C_1, C_1A_1, A_1B_1;
and H, O_a, A_1; H, O_b, B_1; H, O_c, C_1 are collinear.

(24) What has been already proved with regard to the triangle ABC, its orthocentre H, its circumcentre O, and the circles O_a, O_b, O_c may be applied, with the necessary modifications, to the triangle HCB, its orthocentre A, its circumcentre O_a, and the circles O, O_c, O_b; and to the triangles CHA, BAH.

(25) *If RT, RS meet BC at D, D′; SR, ST meet CA at E, E′; TS, TR meet AB at F, F′, then*

$$HDRD',\ HESE',\ HFTF' \text{ are rhombi,}$$

and $$D, H, E';\ E, H, F';\ F, H, D';$$
$$D', H, F;\ E', H, D;\ F', H, E \text{ are collinear.*}$$

FIGURE 42.

Since HR is bisected perpendicularly by DD′,
therefore HD = RD and HD′ = RD′.
But since XY, XZ make equal angles with BC,
and RS, RT are respectively parallel to XY, XZ;
therefore RD = RD′, and HDRD′ is a rhombus.

Again since DH is parallel to RD′
and HE′ ,, ,, ,, ES;
therefore D, H, E′ are collinear.

(26) *If R_1T_1, R_1S_1 meet CB at D, D′; S_1R_1, S_1T_1 meet BH at E′, E; T_1S_1, T_1R_1 meet HC at F′, F, then*

$$ADR_1D',\ AES_1E',\ AFT_1F' \text{ are rhombi,}$$

and D, A, E; E′, A, F; F′, A, D′, etc., are collinear.

FIGURE 40.

Two other triads of rhombi, and of collinear points may be obtained from triangles CHA, BAH.

(27) *If U, V, W be the mid points of AH, BH, CH, then U, V, W are the orthocentres of triangles AC′B′, C′BA′, B′A′C.*

FIGURE 43.

For the perpendicular from B′ to AC′ is parallel to CH;
and since B′ is the mid point of AC, this perpendicular passes through the mid point of AH, that is U;
and AU is perpendicular to C′B′.

(28) *The points U, V, W, H form an orthic tetrastigm, where H is the orthocentre of UVW.*

---

* *Nouvelles Annales*, 2nd series, XIX. 176 (1880) and 3rd series, I. 186–9 (1882).

If the triangle UVW be translated so that U moves along UA and VW remains parallel to BC, it will coincide with triangle AC′B′.

Similarly the triangle UVW may be made to coincide with C′BA′ and B′A′C.

(29) FIGURE 43.

$$\left.\begin{array}{l} U,\ B',\ C' \\ A',\ V,\ C' \\ A',\ B',\ W \end{array}\right\} \text{ are orthocentres of } \left\{\begin{array}{l} HWV,\ CA'W,\ BVA' \\ CWB',\ HUW,\ AB'U \\ BC'V,\ AUC',\ HVU. \end{array}\right.$$

(30) *The point H may be the orthocentre of an infinite number of triangles inscribed in the circle ABC.*

FIGURE 39.

For, take any point A on the circumference;
and draw the chord AHR.
Bisect HR at X, and through X draw the chord BC perpendicular to AR.

Then ABC is a triangle whose orthocentre is H.

(31) *The point A may be the orthocentre of an infinite number of triangles inscribed in the circle HCB.*

FIGURE 40.

For, take any point H on the circumference;
and draw the secant AHR_1.
Bisect AR_1 at X, and through X draw the chord BC perpendicular to AR_1.

Then HCB is a triangle whose orthocentre is A.

Similarly for B and C.

(32) *The straight lines joining the circumcentre to the vertices of a triangle are perpendicular to the sides of the orthic triangle; and the straight lines joining the orthocentre to the vertices are perpendicular to the sides of the complementary triangle; and conversely.*

FIGURE 44.

Let OA meet YZ at X′;
from O draw OB′ perpendicular to CA;
from B′ draw B′C′ parallel to BC.

Then B′ is the mid point of CA,
and B′C′ is a side of the complementary triangle.

Hence $\angle AOB' = \angle ABC = \angle AYX'$;
therefore $\angle AX'Y = \angle AB'O =$ a right angle;
and HA is perpendicular to B′C′.

This theorem will be found to be a particular case of a more general one regarding isogonal lines.

(33) The straight lines joining the circumcentre to the vertices of a triangle are perpendicular to all straight lines which are anti-parallel to the sides with respect to the opposite angles; and the straight lines joining the orthocentre to the vertices are perpendicular to all straight lines which are parallel to the sides.

(34) *The straight lines AX and AO are isogonals * with respect to angle BAC.*

FIGURE 44.

For $\angle ABX = \angle AOB'$,
$\angle AXB = \angle AB'O$;
therefore $\angle XAB = \angle OAC$.

Similarly BY, BO are isogonals with respect to $\angle B$.
and CZ, CO „ „ „ „ „ $\angle C$.

The theorem may be stated and proved otherwise, thus:

The straight lines joining the incentre with the vertices of a triangle bisect the angles between the radii of the circumcircle drawn to the vertices and the perpendiculars.

FIGURE 45.

Produce AI to meet the circumcircle in U, and join OU.

Because AU bisects $\angle BAC$,
therefore U is the mid point of the arc BUC;
therefore OU is perpendicular to BC;
therefore $\angle XAU = \angle OUA = \angle OAU$.

* This corollary is established in the proof of the theorem known as Brahmegupta's.

(35) *The straight lines joining the mid points of*

AH, BC; BH, CA; CH, AB

make with

AB, BC, CA

angles complementary to*

C, A, B.

FIGURE 43.

Let A′, U be the mid points of BC, AH, and O the circumcentre. Join OA.

Then OA′ is equal and parallel to AU;
therefore OA is parallel to A′U.
Now OA makes with AB an angle equal to CAH,
that is, an angle complementary to C;
therefore A′U makes with AB an angle complementary to C.

(36) *The same straight lines make with AX, BY, CZ angles equal to $B \sim C$, $C \sim A$, $A \sim B$.*

For
$$\begin{aligned} \angle A'UX &= \angle OAX \\ &= \angle BAX - \angle CAX \\ &= C - B. \end{aligned}$$

(37) *The angle between†*

$B'Z$ and $C'Y = 3A$, $C'X$ and $A'Z = 3B$, $A'Y$ and $B'X = 3C$.

FIGURE 46.

Produce BY to B_1 so that $B_1Y = BY$,
and " CZ " C_1 " " $C_1Z = CZ$,
and join AB_1, AC_1.

Then $\angle B_1AY$ and $\angle C_1AZ$ are each equal to A;
therefore $\angle B_1AC_1 = 3A$.
But since C′ and Y are the mid points of BA and BB_1,
therefore C′Y is parallel to AB_1.
Similarly B′Z " " " AC_1;

* Dr C. Taylor in *Mathematical Questions from the Educational Times*, XVIII. 65 (1872).

† This property and the demonstration of it are due to Professor R. E. Allardice.

therefore the angle between B′Z and C′Y is equal to the angle between AB_1 and AC_1.

(38) *The straight lines drawn from the orthocentre of a triangle through the mid points of the sides and terminated by the circumcircle are bisected by the sides.*

FIGURE 47.

Let ABC be the triangle, H its orthocentre.

Draw CL parallel to HB and terminated by the circumcircle. Join BL.

Because CL is parallel to AB,

therefore $\angle$ ACL is right;

therefore $\angle$ ABL „ „ ;

therefore BL is parallel to HC.

Hence HBLC is a parallelogram, and its diagonals bisect each other;

that is, HL drawn through A′, the mid point of BC, is bisected by BC.

This corollary may be used to prove part of the characteristic property of the nine-point circle.

(39) $$A'Y = A'Z, \quad B'Z = B'X, \quad C'X = C'Y.$$

FIGURE 47.

For B, Z, Y, C are situated on the circumference of a circle whose centre is A′.

(40) *If on each side of a triangle as diagonal two parallelograms be constructed, the one having a vertex at the opposite angle of the triangle, the other at the centre of the circumcircle, then the straight lines which join the other vertices of these three pairs of parallelograms will pass through the orthocentre.**

FIGURE 48.

FIRST DEMONSTRATION.

Let H be the orthocentre, O the circumcentre; and let O′ and A′ be the vertices opposite to O and A of the parallelograms of which BC is the common diagonal.

* Mr W. J. C. Miller in the *Lady's and Gentleman's Diary* for 1862, p. 74.

Since $\angle BHC$ is supplementary to $\angle A$,
therefore $\angle BHC$ „ „ „ $\angle A'$;
therefore H is on the circumcircle of A'BC.
Now $\angle A'BH$ is right;
therefore A'H is a diameter of the circle A'BC;
therefore A'H passes through O' its centre.

SECOND DEMONSTRATION.*

Draw AA_1 parallel to BC;
join A'O and produce it to meet the circumcircle in R;
join AR meeting BC at X.

Then AR is perpendicular to BC;
and if we imagine the whole figure reflected in BC,
A_1 and O will reflect into the vertices of the two parallelograms on BC as diagonal.
Hence the line joining these vertices will meet AX at the point H, the reflection of R.
But since $\angle BHC = \angle BRC = 180° - \angle BAC$,
therefore H is the orthocentre of ABC;
therefore the straight line joining the two vertices of the parallelogram on BC as diameter passes through the orthocentre.

(41) *If through A, B, C there be drawn AC_1, BA_1, CB_1 making equal angles respectively with HA, HB, HC, a new triangle $A_1B_1C_1$ is formed, which is similar to ABC, and whose circumcentre*† *is H.*

FIGURE 49.

Because $\angle HCB_1 = \angle HBA_1$
therefore the points H, C, A_1, B are concyclic;
therefore $\angle A_1 = 180° - BHC = \angle A$.
Similarly $\angle B_1 = \angle B$, $\angle C_1 = \angle C$.

Join HB_1, HC_1.

* "Conic" of St John's College, Cambridge, in the *Lady's and Gentleman's Diary* for 1863, p. 51.

† C. F. A. Jacobi, *De Triangulorum Rectilineorum Proprietatibus*, p. 34 (1825).

Because	$\angle ACH = \angle ABH$,
and	$\angle ACH = \angle AB_1H$,
	$\angle ABH = \angle AC_1H$;
therefore	$\angle AB_1H = \angle AC_1H$;
therefore	$HB_1 = HC_1$.
Similarly	$HC_1 = HA_1$;

therefore H is the circumcentre of $A_1B_1C_1$.

(42) Since HA, HB, HC are respectively perpendicular to BC, CA, AB, the theorem of the preceding corollary is equivalent to the following:

If through the vertices of a triangle straight lines be drawn making equal angles with the opposite sides, they will form by their intersection a new triangle, which is similar to the original triangle, and which has for circumcentre the orthocentre of the original triangle.

A particular case of this theorem has already been given, that, namely, where the straight lines drawn through the vertices are parallel to the opposite sides. The triangle $A_1B_1C_1$ so formed, the anticomplementary triangle of ABC, is the maximum triangle that can be constructed under such conditions, and it it is equal to four times ABC.

(43) *Triangle XYZ is the triangle of minimum perimeter* inscribed in ABC.*

It is usually considered that this statement is proved † when it is shown that XY and XZ make equal angles with BC

,,	YZ	,,	YX	,,	,,	,,	,, CA
,,	ZX	,,	ZY	,,	,,	,,	,, AB.

No objection can be taken to the following proof ‡:

FIGURE 50.

Produce YZ both ways, making ZX_1 equal to ZX, YX_2 equal to YX; then X_1X_2 is the perimeter of XYZ.

Join BX_1, CX_2.

Because $\angle XZB = \angle AZY = \angle X_1ZB$

* J. F. de Tuschis a Fagnano in *Nova Acta Eruditorum anni* 1775, p. 296.

† See Prof. R. E. Allardice's paper "On a property of odd and even polygons" in the *Proceedings of the Edinburgh Mathematical Society*, VIII. 23 (1890).

‡ Marsano, *Considerazioni sul Triangolo Rettilineo*, pp. 18, 19 (1863).

therefore triangles XZB and X_1ZB are congruent.

Similarly „ XYC „ X_2YC „ „ .

If now DEF be any other triangle inscribed in ABC, and along BX_1 there be taken BD_1 equal to BD, and along CX_2 there be taken CD_2 equal to CD, and FD_1, ED_2 be joined, it may be proved that $FD_1 = FD$, $ED_2 = ED$, and that consequently the line D_1FED_2 is the perimeter of DEF.

If D_1FED_2 is not straight, join D_1D_2 and join the vertex A with X, D, X_1, D_1, X_2, D_2.

Then $$AX_1 = AX = AX_2,$$
$$AD_1 = AD = AD_2;$$

therefore the triangles AX_1X_2, AD_1D_2 are isosceles.
And their vertical angles X_1AX_2, D_1AD_2 are equal,
since each is double of angle BAC;
therefore the triangles AX_1X_2, AD_1D_2 are similar.
Now AX_1 is less than AD_1, since AX is less than AD;
therefore X_1X_2 is less than D_1D_2, and consequently less than D_1FED_2.

If the triangle ABC be right-angled at A, the points Y, Z coalesce with A, X_1X_2 and D_1D_2 pass through A and are respectively double of AX and AD.

If the triangle ABC be obtuse-angled at A, the points Y, Z fall outside the triangle ABC (*Figure* 51) and X_1X_2 is now equal to $XY - YZ + ZX$. If therefore the preceding statements and proof are to hold good, the side YZ must be considered negative.

(44) *If XX_1, XX_2 be joined cutting AB, AC in P, Q, then PQ is the semiperimeter* of triangle XYZ.*

FIGURES 50, 51.

For P is the mid point of XX, and Q the mid point of XX_2;

therefore $$PQ = \tfrac{1}{2}X_1X_2 = \text{semiperimeter of } XYZ.$$

P and Q are the feet of the perpendiculars from X on AB and AC.

If triangle ABC be obtuse-angled, the perimeter of XYZ must be understood with the qualification of the preceding corollary.

* Lhuilier, *Élémens d'Analyse*, p. 231 (1809). The proof in the text is given by Feuerbach, *Eigenschaften...des...Dreiecks*, Section VI., Theorem 8 (1822).

(45) *If two triangles ABC, $A'B'C'$ have their sides parallel, and one of them is circumscribed about and the other is inscribed in the same triangle DEF, the area of this last triangle is a mean proportional between the areas of the two others.**

FIGURE 52.

Let AB′, AC′ meet BC at P and Q. Through A′ draw A′A″ parallel to B′C′ or BC and meeting AC′ at A″.
Join A″B′, AA′, B′Q.

Then $A'B'F = A'B'A$, $A'C'E = A'C'A$, $B'C'D = B'C'Q$;

therefore $\qquad DEF = AB'Q$, $A'B'C' = A''B'C'$.

Now $\qquad A''B'C' : AB'Q = A''C' : AQ$;

and $A''C' : AQ$ is the ratio of the altitudes of the similar triangles $A'B'C'$, ABC.

Hence $\qquad A''C' : AQ = B'C' : BC$;

therefore $\qquad A''B'C' : AB'Q = B'C' : BC$.

Again $\qquad AB'Q : APQ = AB' : AP$
$$= B'C' : PQ;$$

and $\qquad APQ : ABC = PQ : BC$;

therefore $\qquad AB'Q : ABC = B'C' : BC$;

therefore $\qquad A''B'C' : AB'Q = AB'Q : ABC$

or $\qquad A'B'C' : DEF = DEF : ABC$.

The terms *inscribed* and *circumscribed* have the following signification,

One triangle is inscribed in a second triangle when the vertices of the first are situated on the sides or the sides produced of the second; and in either case the second triangle is circumscribed about the first.

---

* This theorem is due to Mr Rochat of Saint-Brieux, and is thus stated in Gergonne's *Annales de Mathématiques* II. 93 (1811-2).

If to any triangle T there be circumscribed another T', and to T' a third T'' having its sides respectively parallel to those of T; then to T'' a new triangle T''' having its sides respectively parallel to those of T', and so on: the triangles T, T', T'', T''' will be similar in pairs and form a geometrical progression.

The demonstration in the text is given by Mr Léon Anne, in the *Nouvelles Annales*, III. 27 (1844).

(46) *If $I_1I_2I_3$ be the fundamental triangle, ABC its orthic triangle, and DEF the triangle formed by joining the points of contact of the incircle of ABC, then**

$$I_1I_2I_3 : ABC = ABC : DEF.$$

FIGURE 28.

In the same way if II_3I_2 be the fundamental triangle, ABC its orthic triangle, and $D_1E_1F_1$ the triangle formed by joining the points of contact of the first excircle of ABC, then

$$II_3I_2 : ABC = ABC : D_1E_1F_1;$$

and so on.

(47)
$$ABC : DEF = 2R : r$$
$$ABC : D_1E_1F_1 = 2R : r_1$$

and so on.

For
$$(ABC)^2 : (DEF)^2 = I_1I_2I_3 : DEF$$
$$= 4R^2 : r^2$$

since 2R and r are the radii of the circumcircles of the similar triangles $I_1I_2I_3$ and DEF.

(48) *If ABC be the fundamental triangle, DEF the triangle formed by joining the points of contact of the incircle of ABC, and X'Y'Z' the orthic triangle of DEF, then*

$$ABC : DEF = DEF : X'Y'Z'.$$

FIGURE 53.

For $\angle BDF = \angle DEF = \angle DY'Z'$;
therefore Y'Z' is parallel to BC.
Hence Z'X' „ „ CA
and X'Y' „ „ AB.

In the same way if ABC be the fundamental triangle, $D_1E_1F_1$ the triangle formed by joining the points of contact of the first excircle of ABC, and the orthic triangle of $D_1E_1F_1$ be constructed, it will be found that this orthic triangle has its sides parallel to those of ABC, and that $D_1E_1F_1$ is a mean proportional between it and ABC.

* The theorems (46)-(48) are given by Feuerbach, *Eigenschaften...des...Dreiecks*, §§ 61, 8, 63 (1822).

(49) Hence $I_1I_2I_3$, ABC, DEF, X'Y'Z', ...
are a series of triangles whose areas form a geometrical progression, the alternate terms being similar.

Other series may be obtained from

II_3I_2, ABC, $D_1E_1F_1$, ..., etc.

(50) $$ABC : X'Y'Z' = 4R^2 : r^2.$$

Def. If P be any point in the plane of ABC, and D, E, F be the projections of P on BC, CA, AB, then DEF is called the *pedal* triangle of P with respect to ABC.

(51) *If* H_1 , H_2 , H_3
be the orthocentres of the triangles

AEF , *BFD* , *CDE*

cut off from ABC by the sides of the pedal triangle DEF of any point P, the triangle $H_1H_2H_3$ is congruent and oppositely situated to DEF.

Figure 54.

Since	PD, FH_2 are perpendicular to BC,
therefore	PD is parallel to FH_2.
Similarly	PF ,, ,, DH_2;
therefore	PDH_2F is a parallelogram,
and	$PD = FH_2$.
Hence also	$PD = EH_3$,
therefore	EFH_2H_3 is a parallelogram,
and	$H_2H_3 = EF$.

Figure 55.

The sides of the four triangles

DEF, $D_1E_1F_1$, $D_2E_2F_2$, $D_3E_3F_3$

make with the sides of ABC the following

twelve triangles	whose orthocentres are
AEF , BFD , CDE	H_1, H_2, H_3
AE_1F_1, BF_1D_1, CD_1E_1	H_1', H_2', H_3'
....	....
....	....

(52) The twelve orthocentres are situated in pairs on the six lines

$$\mathrm{II_1,\ II_2,\ II_3,\ I_2I_3,\ I_3I_1,\ I_1I_2}.$$

(53) The four triangles

$$\mathrm{H_1H_2H_3,\quad H_1'H_2'H_3'},\ \text{and so on},$$

are congruent and oppositely situated to

$$\mathrm{D\ E\ F,\quad D_1\ E_1\ F_1},\ \text{and so on}.$$

(54) The following figures are rhombi:

$\mathrm{DH_3EI}$,	$\mathrm{EH_1FI}$,	$\mathrm{FH_2DI}$
$\mathrm{D_1H_3'E_1I_1}$,	$\mathrm{E_1H_1'F_1I_1}$,	$\mathrm{F_1H_2'D_1I_1}$
....	....	....
....	....	....

their sides being r, r_1, r_2, r_3 respectively.

(55) The following figures are equilateral hexagons:

$$\mathrm{DH_3EH_1FH_2,\ D_1H_3'E_1H_1'F_1H_2'},\ \ldots\ldots\ldots\ldots,\ \ldots\ldots\ldots\ldots$$

their perimeters being $6r$, $6r_1$, $6r_2$, $6r_3$ respectively.

(56) I, I_1, I_2, I_3

which are the circumcentres of the triangles

$$D\ E\ F,\quad D_1\ E_1\ F_1,\quad \textit{and so on},$$

*are the orthocentres of the triangles**

$$H_1H_2H_3,\quad H_1'H_2'H_3',\quad \textit{and so on}.$$

Take, for example, the triangle $\mathrm{H_1H_2H_3}$.

Because $\mathrm{H_1I}$ is perpendicular to EF,

therefore $\mathrm{H_1I}$ „ „ „ $\mathrm{H_2H_3}$.

Similarly for $\mathrm{H_2I}$ and $\mathrm{H_3I}$.

(57) *If* H_0, H_0', H_0'', H_0'''

be the orthocentres of the triangles

$$D\ E\ F,\quad D_1\ E_1\ F_1,\quad \textit{and so on},$$

they will be the circumcentres of the triangles *

$$H_1H_2H_3,\quad H_1'H_2'H_3',\quad \textit{and so on}.$$

* The first parts of (56) and (57) are given by Feuerbach, *Eigenschaften...des... Dreiecks*, §§ 87, 88 (1822).

FIGURE 56.

Take, for example, the triangle $H_1H_2H_3$.

Because DH_0 is perpendicular to EF,

therefore DH_0 ,, ,, H_2H_3.

And since $DH_2 = DH_3$,

therefore DH_0 bisects H_2H_3

therefore DH_0 passes through the circumcentre of $H_1H_2H_3$

Similarly for EH_0 and FH_0.

(58)

$$H_0H_1 = H_0H_2 = H_0H_3$$
$$= I\ D = I\ E = I\ F = r$$

For H_0 and I are the circumcentres of two congruent triangles $H_1H_2H_3$ and DEF.

Similarly for H_0', I_1, and so on.

(59) The following figures are parallelograms:

$$DIH_1H_0,\ EIH_2H_0,\ FIH_3H_0;$$

they have a common diagonal IH_0;
their other diagonals intersect at the mid point of IH_0.

Similarly for I_1H_0', and so on.

(60) The homothetic centre of DEF, $H_1H_2H_3$ is the mid point of IH_0.

Similarly for $D_1E_1F_1$, $H_1'H_2'H_3'$, and so on.

(61) The following figures are rhombi:

$$DH_3H_0H_2,\ EH_1H_0H_3,\ FH_2H_0H_1,$$

and their sides are equal to r.

Three other triads of rhombi can be obtained by putting subscripts and accents to the preceding letters.

(62) If from H_1, H_2, H_3 perpendiculars be drawn to BC, CA, AB, these perpendiculars will be concurrent at H_0.

Since $EH_1H_0H_3$ is a rhombus,

therefore H_1H_0 is parallel to EH_3

therefore H_1H_0 is perpendicular to BC.

Similarly for H_2H_0 and H_3H_0.

(63) Since I and H_0 are the circumcentre and orthocentre of DEF and the orthocentre and circumcentre of $H_1H_2H_3$, these two triangles have the same nine-point circle,* and its centre is the mid point of IH_0.

FIGURE 56.

(64) In triangle DEF

$$\left.\begin{matrix} DI, & DH_0 \\ EI, & EH_0 \\ FI, & FH_0 \end{matrix}\right\} \text{ are isogonals with respect to } \left\{\begin{matrix} D \\ E \\ F \end{matrix}\right.$$

(65) In triangle $H_1H_2H_3$

$$\left.\begin{matrix} H_1I, & H_1H_0 \\ H_2I, & H_2H_0 \\ H_3I, & H_3H_0 \end{matrix}\right\} \text{ are isogonals with respect to } \left\{\begin{matrix} H_1 \\ H_2 \\ H_3 \end{matrix}\right.$$

(66) *Of the perpendiculars to BC from* H_1, H_2, H_3, *I the first is equal to the sum of the other three.*†

FIGURE 57.

Let the feet of the perpendiculars on BC from H_1, H_2, H_3 be X_1, X_2, X_3;

let IH_1 meet EF at D′;

from D′ draw a perpendicular to BC, meeting BC at D″ and H_2H_3 at L.

Then D′ is the mid point of EF and IH_1;

therefore D″ „ „ „ „ „ X_2X_3 „ X_1D.

Also L „ „ „ „ „ H_2H_3;

therefore L „ „ „ „ „ DH_0,

since $DH_3H_0H_2$ is a rhombus.

Hence $H_2X_2 + H_3X_3 = 2LD'' = H_0X_1$;

and $ID = H_0H_1$;

therefore $H_2X_2 + H_3X_3 + ID = H_1X_1$.

---

* Feuerbach, § 89.

† Feuerbach, § 80. The mode of proof is not his.

In triangle ABC, the perpendiculars AX, BY, CZ intersect at H the orthocentre, and XYZ is the orthic triangle.

FIGURE 58.

This figure has reference to the properties (67)–(82). The reader would find it convenient if he constructed a copy of it on a large scale.

Of the triangles	let the orthocentres be
AYZ, XBZ, XYC	H_1 , H_2 , H_3
HYZ, XCZ, XYB	H_1' , H_2' , H_3'
CYZ, XHZ, XYA	H_1'', H_2'', H_3''
BYZ, XAZ, XYH	H_1''', H_2''', H_3'''

(67) Of these H points, four pairs are collinear with X, four with Y, and four with Z, that is, through

X pass H_2H_2''', H_3H_3'', $H_2'H_2''$, $H_3'H_3'''$
Y ,, H_3H_3', H_1H_1''', $H_3''H_3'''$, $H_1'H_1''$.
Z ,, H_1H_1'', H_2H_2', $H_1'H_1'''$, $H_2''H_2'''$

(68) The four* triangles $H_1H_2H_3$, $H_1'H_2'H_3'$ etc., are congruent and oppositely situated to XYZ.

(69) *The three triangles $H_1'H_1''H_1'''$, $H_2'H_2''H_2'''$, $H_3'H_3''H_3'''$ are congruent and oppositely situated to ABC; and H_1, H_2, H_3 are their respective orthocentres.*

Take for example $H_3'H_3''H_3'''$.

Because $H_3''H_3'''$ passes through Y and is perpendicular to AX,
therefore $H_3''H_3'''$ is parallel to BC.
Similarly $H_3'''H_3'$ is parallel to CA.
Again $H_2'H_3'$ is equal and parallel to $H_2''H_3''$;
therefore $H_3'H_3''$,, ,, ,, ,, ,, $H_2'H_2''$.
Now $H_2'H_2''$ passes through X and is perpendicular to CZ;
therefore $H_3'H_3''$ is parallel to AB.
Hence $H_3'H_3''H_3'''$ is similar and oppositely situated to ABC.

* Feuerbach (§ 90) proves the congruency of XYZ, $H_1H_2H_3$

Because $H_3'H_3$ passes through Y and is perpendicular to BC,
and $H_3''H_3$ „ „ X „ „ „ „ CA;
therefore Y, X are the feet of two of the perpendiculars,
and H_3 is the orthocentre, of triangle $H_3'H_3''H_3'''$.

Lastly, since H, X in triangle ABC
correspond to H_3, Y „ „ $H_3'H_3''H_3'''$, and $HX = H_3Y$,
therefore triangles ABC, $H_3'H_3''H_3'''$ are congruent.

(70) *If* $H_1'H_1$ *meet* $H_1''H_1'''$ *at* X_1,
$H_2''H_2$ „ $H_2'''H_2'$ „ Y_1,
$H_3'''H_3$ „ $H_3'H_3''$ „ Z_1;

then the feet of the perpendiculars of triangle

$H_1'H_1''H_1'''$ *are* X_1, Z, Y,
$H_2'H_2''H_2'''$ „ Z, Y_1, X,
$H_3'H_3''H_3'''$ „ Y, X, Z_1;

and the sides of triangle $X_1Y_1Z_1$ *pass through* X, Y, Z *and are there bisected.*

Because triangles $H_1'H_1''H_1'''$ and ABC are congruent and oppositely situated,
therefore their orthic triangles X_1ZY and XYZ are congruent and oppositely situated.
Similarly ZY_1X and YXZ_1 are congruent and oppositely situated to XYZ;
therefore Y_1Z_1 passes through X and is bisected at X
Z_1X_1 „ „ Y „ „ „ „ Y
X_1Y_1 „ „ Z „ „ „ „ Z.

(71) *ABC*, $X_1Y_1Z_1$ *have the same nine-point circle.*

For X, Y, Z, the feet of the perpendiculars of ABC, are the mid points of the sides of $X_1Y_1Z_1$.

(72) *If* O, O_a, O_b, O_c *be the circumcentres of*

ABC, *HCB*, *CHA*, *BAH*;

then the point of concurrency of

	AH_1 ,	BH_2 ,	CH_3	*is*	O
	HH_1' ,	CH_2' ,	BH_3'	,,	O_a
	CH_1'' ,	HH_2'' ,	AH_3''	,,	O_b
	BH_1''',	AH_2''',	HH_3'''	,,	O_c
and	O ,	O_a ,	O_b ,	O_c	*are orthocentres*
of	$H_1H_2H_3$,	$H_1'H_2'H_3'$,	$H_1''H_2''H_3''$,	$H_1'''H_2'''H_3'''$.	

For AH_1, BH_2, CH_3 are respectively perpendicular
to YZ, ZX, XY;
and their concurrency is established by Steiner's theorem concerning orthologous triangles. See § 6 (1).

Since AH_1. BH_2, CH_3 are respectively perpendicular
to YZ, ZX, XY, they are therefore perpendicular
to H_2H_3, H_3H_1, H_1H_2,
and consequently concurrent at the orthocentre of $H_1H_2H_3$.

(73) *If the homothetic centre of the triangles*

XYZ	*and*	$H_1\ H_2\ H_3$	*be*	T
XYZ	,,	$H_1'\ H_2'\ H_3'$	,,	T_1
XYZ	,,	$H_1''\ H_2''\ H_3''$	,,	T_2
XYZ	,,	$H_1'''H_2'''H_3'''$	,,	T_3 ,

then $T_1T_2T_3$ is similar and oppositely situated to ABC, and T, T_1, T_2, T_3 form an orthic tetrastigm.

For T_2 is the mid point of XH_1''
T_3 ,, ,, ,, ,, ,, XH_1''';
therefore T_2T_3 is parallel to $H_1''H_1'''$ and equal to half of it;
therefore T_2T_3 ,, ,, ,, BC ,, ,, ,, ,, ,, ,, .

Again T is the mid point of XH_1
T_1 ,, ,, ,, ,, ,, XH_1';
therefore TT_1 is parallel to H_1H_1' and equal to half of it;
therefore TT_1 is perpendicular to $H_1''H_1'''$ or to T_2T_3,
and T is orthocentre of $T_1T_2T_3$.

(74) *The point T is the centre of the three parallelograms*

$$YZH_2H_3,\quad ZXH_3H_1,\quad XYH_1H_2.$$

For YH_2, ZH_3 intersect at T.

Similarly T_1, T_2, T_3 are each the centre of three parallelograms.

(75) *If X', Y', Z' be the mid points of YZ, ZX, XY, then the point of concurrency of*

$H_1\ X'$,	$H_2\ Y'$,	$H_3\ Z'$	*is*	H
$H_1'\ X'$,	$H_2'\ Y'$,	$H_3'\ Z'$	,,	A
$H_1''\ X'$,	$H_2''\ Y'$,	$H_3''\ Z'$	,,	B
$H_1'''X'$,	$H_2'''Y'$,	$H_3'''Z'$	,,	C.

For YH_1 and HZ are parallel, and so are ZH_1 and HY; therefore HYH_1Z is a parallelogram; therefore HH_1 and YZ bisect each other, that is, H_1X' passes through H.

Again, ABC, $H_1'H_1''H_1'''$ are congruent and oppositely situated, and Y in ABC corresponds to Z in $H_1'H_1''H_1'''$; therefore $AYH_1'Z$ is a parallelogram; therefore AH_1' and YZ bisect each other, that is, $H_1'X'$ passes through A.

(76) Let the incircle and excircles of XYZ be denoted by their centres H, A, B, C; then the radical axes of

	H, A;	H, B;	H, C;	B, C;	C, A;	A, B
are	$T_2\ T_3$,	$T_3\ T_1$,	$T_1\ T_2$,	$T_1\ T$,	$T_2\ T$,	$T_3\ T$.

For T_2T_3 is perpendicular to HA, and bisects YZ.

(77) The circles A, B, C; H, C, B; C, H, A; B, A, H have T, T_1; T_2, T_3 for radical centres.

(78) *X', Y' Z' are the feet of the perpendiculars of $T_1T_2T_3$.*

For in triangle AXH_1' the mid point of XH_1' is T_1, and T_1T is parallel to AX; therefore T_1T passes through X', the mid point of AH_1'.

Hence T, T_1, T_2, T_3 are the incentre and the excentres of the triangle $X'Y'Z'$. Compare § 5, (11), (12).

(79) The homothetic centre of the triangles

$$\begin{array}{llll} T_1T_2T_3 & \text{and} & H_1'H_1''H_1''' & \text{is } X \\ T_1T_2T_3 & ,, & H_2'H_2''H_2''' & ,, \ Y \\ T_1T_2T_3 & ,, & H_3'H_3''H_3''' & ,, \ Z. \end{array}$$

For T_2, T_3 are mid points of XH_1'', XH_1'''.

(80) Since $X'Y'Z'$ is the complementary triangle of XYZ,
and T, T_1, T_2, T_3 are the incentre and excentres of $X'Y'Z'$,
and H, A, B, C ,, ,, ,, ,, ,, ,, XYZ;
therefore HT, AT_1, BT_2, CT_3 all pass through the centroid of XYZ. See § 2.

If G' denote this centroid *

then $$HG' : TG' = AG' : T_1G' = BG' : T_2G' = CG' : T_3G' = 2 : 1.$$

(81) Since H is the incentre, G' the centroid, of XYZ, and T the incentre of $X'Y'Z'$, if $HG'T$ be produced to J' so that $TJ' = HT$, then J' will be the incentre of $X_1Y_1Z_1$.

Similarly J_1', J_2', J_3', situated on AT_1, BT_2, CT_3, so that $T_1J_1' = AT_1$ and so on, will be the first, second, and third excentres of $X_1Y_1Z_1$.

These statements follow from the first few corollaries of § 2.

(82) The tetrads of points

H, G', T, J'; A, G', T_1, J_1'; B, G', T_2, J_2'; C, G', T_3, J_3'

form harmonic ranges.

(83) Since triangles $I_1I_2I_3$, ABC stand to each other in the same relation as ABC, XYZ, the second being the orthic triangle of the first, it may be convenient to state in another form some of the results already established.

The means of transliteration from the one form to the other will be afforded by the following lists of corresponding points.

---

* G' would naturally denote the centroid of triangle $A'B'C'$, but G is the centroid both of ABC and $A'B'C'$.

$$A,\ B,\ C,\ H,\ X,\ Y,\ Z,\ H_1,\ H_2,\ H_3$$

correspond to

$$I_1,\ I_2,\ I_3,\ I,\ A,\ B,\ C,\ H_a,\ H_b,\ H_c$$

and

$$O,\ O_a,\ O_b,\ O_c,\ G',\ T,\ T_1,\ T_2,\ T_3$$

correspond to

$$O_0,\ O_1,\ O_2,\ O_3,\ G,\ L,\ L_1,\ L_2,\ L_3$$

and

$$X',\ Y',\ Z',\ X_1,\ Y_1,\ Z_1,\ J',\ J_1',\ J_2',\ J_3'$$

correspond to

$$A',\ B',\ C',\ A_1,\ B_1,\ C_1,\ J,\ J_1,\ J_2,\ J_3.$$

Hence the following results* are obtained:

(*a*) The orthocentres of the triangles BCI_1, CAI_2, ABI_3 form the vertices of a triangle $H_aH_bH_c$ which is congruent to the fundamental triangle ABC, and has its sides parallel to the corresponding sides of ABC.

(*b*) AH_a, BH_b, CH_c are concurrent at the radical centre of I_1, I_2, I_3.

(*c*) The radical centre bisects AH_a, BH_b, CH_c.

(*d*) The radical axes of the I circles bisect the sides of $H_aH_bH_c$.

(*e*) O_0 is the orthocentre of $H_aH_bH_c$.

(*f*) AD_1, BE_2, CF_3 are concurrent at J the incentre of $H_aH_bH_c$. The points J, I, L are collinear.

(*g*) H_a, A′, I ; H_b, B′, I ; H_c, C′, I are collinear.

(*h*) A′, B′, C′ bisect H_aI, H_bI, H_cI.

FIGURE 59.

In triangle ABC, the points H, X, Y, Z

are the orthocentre and feet of the perpendiculars;

the various I, D, E, F points are the centres and points of contact of the incircle and the excircles.

The rest of the notation will be explained as it is wanted.

* See Professor Johann Döttl's *Neue merkwürdige Punkte des Dreiecks*, pp. 40-46 (no date). The proofs given in this noteworthy pamphlet are analytical.

(84) The perpendicular AX contains the intersection of

D_2E_2,	D_3F_3	namely	X_0
D_3E_3,	D_2F_2	,,	X_1
$D\,E$,	D_1F_1	,,	X_2
D_1E_1,	$D\,F$	,,	X_3.

The perpendicular BY contains the intersection of

E_3F_3,	E_1D_1	namely	Y_0
E_2F_2,	$E\,D$	,,	Y_1
E_1F_1,	E_3D_3	,,	Y_2
$E\,F$,	E_2D_2	,,	Y_3.

The perpendicular CZ contains the intersection of

F_1D_1,	F_2E_2	namely	Z_0
$F\,D$,	F_3E_3	,,	Z_1
F_3D_3,	$F\,E$	,,	Z_2
F_2D_2,	F_1E_1	,,	Z_3.

FIGURE 60.

Through A draw a parallel to BC;
let D_2E_2 meet AX at X_0 and the parallel at S.

Then triangles CD_2E_2, ASE_2 are similar;

and because $\quad CD_2 = CE_2$

therefore $\quad AS = AE_2 = s_3$.

Now triangles AX_0S, DIC have their sides respectively parallel to each other; therefore they are similar.

But $\quad AS = s_3 = DC$;

therefore $\quad AX_0 = DI = r$.

Again if D_3F_3 meet the parallel through A at T, and AX at X_0', it may be proved that

$$AT = AF_3 = s_2 = DB$$

and that triangles $AX_0'T$, DIB are congruent;

therefore. $\quad AX_0' = DI = r$,

and X_0, X_0' are the same point.

The other properties are proved in a manner exactly analogous.

(85) FIGURE 59.

$$AX_0 = BY_0 = CZ_0 = r$$
$$AX_1 = BY_1 = CZ_1 = r_1$$
$$AX_2 = BY_2 = CZ_2 = r_2$$
$$AX_3 = BY_3 = CZ_3 = r_3.$$

Attention may be directed here and later on to the way in which the various suffixes occur.

$$\text{The triads of lines} \quad \left.\begin{array}{lll} E_1F_1, & F_2D_2, & D_3E_3 \\ E\,F, & F_3D_3, & D_2E_2 \\ E_3F_3, & F\,D, & D_1E_1 \\ E_2F_2, & F_1D_1, & D\,E \end{array}\right\} \text{ determine the triangles } \left\{\begin{array}{l} X_1Y_2Z_3 \\ X_0Y_3Z_2 \\ X_3Y_0Z_1 \\ X_2Y_1Z_0 \end{array}\right.$$

(86) The four triangles

$$X_1Y_2Z_3, \quad X_0Y_3Z_2, \quad X_3Y_0Z_1, \quad X_2Y_1Z_0$$

are respectively similar and oppositely situated to

$$I_1\ I_2\ I_3, \quad I\ I_3\ I_2, \quad I_3\ I\ I_1, \quad I_2\ I_1\ I$$

and H, the orthocentre of ABC, is the circumcentre of the four.

Since Y_2Z_3 is perpendicular to AI_1,
therefore Y_2Z_3 is parallel to I_2I_3.
Similarly for Z_3X_1 and X_1Y_2.

Again $\angle HY_2Z_3 = \angle CAI_1$
because the sides of the one are perpendicular to those of the other;
and $\angle HZ_3Y_2 = \angle BAI_1$, for a similar reason;
therefore $\angle HY_2Z_3 = \angle HZ_3Y_2$;
therefore $HY_2 = HZ_3$.
Similarly $HZ_3 = HX_1$;
therefore H is the circumcentre of $X_1Y_2Z_3$.

(87) The radii of the circumcircles of

$$X_1Y_2Z_3, \quad X_0Y_3Z_2, \quad X_3Y_0Z_1, \quad X_2Y_1Z_0$$

are $\quad 2R + r, \quad 2R - r_1, \quad 2R - r_2, \quad 2R - r_3.$

For

$$\begin{aligned} & HX_1 + HY_2 + HZ_3 \\ &= AX_1 + BY_2 + CZ_3 + HA + HB + HC \\ &= r_1 + r_2 + r_3 + 2(k_1 + k_2 + k_3) \\ &= 4R + r + 2r + 2R \\ &= 6R + 3r. \end{aligned}$$

(88) $X_0D = AI$, $X_1D_1 = AI_1$, $X_2D_2 = AI_2$, $X_3D_3 = AI_3$
$Y_0E = BI$, $Y_1E_1 = BI_1$, $Y_2E_2 = BI_2$, $Y_3E_3 = BI_3$
$Z_0F = CI$, $Z_1F_1 = CI_1$, $Z_2F_2 = CI_2$, $Z_3F_3 = CI_3$

Because AX_0 is equal and parallel to ID,
therefore $AIDX_0$ is a parallelogram;
therefore $X_0D = AI$.

(89) In the four pairs of triangles

$$X_1Y_2Z_3, \quad X_0Y_3Z_2, \quad X_3Y_0Z_1, \quad X_2Y_1Z_0$$
$$I_1I_2I_3, \quad I\,I_3I_2, \quad I_3I\,I_1, \quad I_2I_1I$$

consider the intersections of the sides.*

Y_2Z_3	intersects	I_1I_2,	I_1I_3	at	V_1, W_1
Z_3X_1	,,	I_2I_3,	I_2I_1	,,	W_2, U_2
X_1Y_2	,,	I_3I_1,	I_3I_2	,,	U_3, V_3
Y_3Z_2	,,	II_3,	II_2	,,	V', W'
Z_2X_0	,,	I_3I_2,	I_3I	,,	W_2, U''
X_0Y_3	,,	I_2I,	I_2I_3	,,	U''', V_3
Y_0Z_1	,,	I_3I,	I_3I_1	,,	V', W_1
Z_1X_3	,,	II_1,	II_3	,,	W'', U''
X_3Y_0	,,	I_1I_3,	I_1I	,,	U_3, V'''
Y_1Z_0	,,	I_2I_1,	I_2I	,,	V_1, W'
Z_0X_2	,,	I_1I,	I_1I_2	,,	W'', U_2
X_2Y_1	,,	II_2,	II_1	,,	U''', V'''.

It will be seen that several theorems are embedded in the preceding notation.

* The notation here is somewhat complicated, but it could not well be otherwise. I have made various attempts to simplify it, but with little success; what is gained in one respect is lost in another.

(90) The sides of the four triangles

$$DEF, \quad D_1E_1F_1, \quad D_2E_2F_2, \quad D_3E_3F_3$$

contain each four other points of the diagram.

EF	contains	Y_3	Z_2	V'	W'
FD	,,	Z_1	X_3	W''	U''
DE	,,	X_2	Y_1	U'''	V'''
E_1F_1	,,	Y_2	Z_3	V_1	W_1
F_1D_1	,,	Z_0	X_2	W'''	U_2
D_1E_1	,,	X_3	Y_0	U_3	V'''
E_2F_2	,,	Y_1	Z_0	V_1	W'
F_2D_2	,,	Z_3	X_1	W_2	U_2
D_2E_2	,,	X_0	Y_3	U'''	V_3
E_3F_3	,,	Y_0	Z_1	V'	W_1
F_3D_3	,,	Z_2	X_0	W_2	U''
D_3E_3	,,	X_1	Y_2	U_3	V_3

(91) The twelve EF, FD, DE lines determine, by their intersections with the six lines of the orthic tetrastigm $II_1I_2I_3$, pairs of feet of the perpendiculars of the triangles

$$\begin{array}{lll} I_1BC\,, & AI_2C\,, & ABI_3 \\ I\,BC\,, & AI_3C\,, & ABI_2 \\ I_3BC\,, & AI\,C\,, & ABI_1 \\ I_2BC\,, & AI_1C\,, & ABI\,. \end{array}$$

The other twelve feet are the various D, E, F points.

It may be useful to remember that these four triads of triangles are similar to

$$I_1I_2I_3\,, \quad I\,I_3I_2\,, \quad I_3I\,I_1\,, \quad I_2I_1I\,.$$

The following proof of one of these properties may be sufficient:

Because $$CD_1 = CE_1$$

therefore triangles CD_1V_1, CE_1V_1 are congruent

and $$\angle CD_1V_1 = \angle CE_1V_1 = \tfrac{1}{2}(B+C)$$
$$= \angle I_3AB\,.$$

Now triangle I_1BC is similar to $I_1I_2I_3$,
the sides I_1B, I_1C, BC being homologous to
" " I_1I_2, I_1I_3, I_2I_3
and I_1D_1 being homologous to I_1A,
so that D_1 and A are homologous points;
therefore V_1 " B " " " ,
since $\angle CD_1V_1 = \angle I_3AB$.
But B is the foot of the perpendicular on I_1I_3 from I_2;
therefore V_1 " " " " " " " I_1C " B.

(92) The following quartets of points form orthic tetrastigms:

$$X_0X_1D_2D_3;\quad Y_0Y_2E_3E_1;\quad Z_0Z_3F_1F_2$$
$$X_2X_3D\,D_1;\quad Y_3Y_1E\,E_2;\quad Z_1Z_2F\,F_3.$$

(93) Through the mid point of

BC	pass	X_0I,	X_1I_1,	X_2I_2,	X_3I_3
CA	"	Y_0I,	Y_1I_1,	Y_2I_2,	Y_3I_3
AB	"	Z_0I,	Z_1I_1,	Z_2I_2,	Z_3I_3.

Take* for example X_1I_1.

Triangles $X_1D_2D_3$, I_1BC are similar and oppositely situated; therefore X_1I_1 passes through their centre of similitude.
But D_3C " " " " " " ;
therefore if X_1I_1 cut D_3C at A', the centre of similitude is A'.

Hence
$$\frac{A'B}{A'C} = \frac{A'D_2}{A'D_3}$$
$$= \frac{A'B + A'D_2}{A'C + A'D}$$
$$= \frac{BD_2}{CD_3} = \frac{s}{s};$$

therefore A' is the mid point of BC.

A shorter demonstration of this would be obtained if (95) were proved before (93).

* This method of proof is due to Professor Neuberg.

For $X_1U_3I_1U_2$ is a parallelogram ;

therefore X_1I_1 bisects U_3U_2 ;

therefore „ „ BC.

(94) The four centres of homology of the four pairs of triangles $X_1Y_2Z_3$, $I_1I_2I_3$, and so on, are the symmedian points of these pairs of triangles.

For I_1X_1 bisects BC,

and BC is antiparallel to I_2I_3 with respect to $\angle I_1$;

therefore I_1X_1 is a symmedian of $I_1I_2I_3$.

Since X_1I_1 bisects BC, it must also bisect D_2D_3.

Now D_2D_3 is antiparallel to Y_2Z_3 with respect to $\angle X_1$;

therefore X_1I_1 is a symmedian of $X_1Y_2Z_3$.

(95) All the U points are on a line parallel to BC

„ „ V „ „ „ „ „ „ „ CA

„ „ W „ „ „ „ „ „ „ AB.

(96)
$$U_2U_3 = V_3V_1 = W_1W_2 = s$$
$$U''U''' = V_3V' = W'W_2 = s_1$$
$$U''U_3 = V'''V' = W_1W'' = s_2$$
$$U_2U''' = V'''V_1 = W'W'' = s_3.$$

Because D_2U_2 is parallel to BU_3,

and CU_2 „ „ „ D_3U_3,

and $CD_2 = s_1 = BD_3$;

therefore $D_2U_2 = BU_3$;

therefore $U_2U_3BD_2$ is a parallelogram ;

therefore U_2U_3 is parallel and equal to BD_2, that is to s.

Similarly the other UU lines are parallel to BC ;

therefore the U points are collinear.

The U points lie on the line $B'C'$.

(97) The following sets of six points are concyclic

$$\begin{array}{cccccc} U_2, & U_3, & V_3, & V_1, & W_1, & W_2 \\ U'', & U''', & V_3, & V', & W', & W_2 \\ U'', & U_3, & V''', & V', & W_1, & W'' \\ U_2, & U''', & V''', & V_1, & W', & W''. \end{array}$$

Because D_2V_3, D_3W_2 are two of the perpendiculars of $X_1D_2D_3$;
herefore W_2V_3 is antiparallel to D_2D_3 with respect to $\angle X_1$;
herefore W_2V_3 is antiparallel to U_2U_3;
herefore W_2, V_3, U_3, U_2 are concyclic.
Similarly U_3, W_1, V_1, V_3 ,, ,,
and V_1, U_2, W_2, W_1 ,, ,, .
Hence all the six points are concyclic.

The four circles are the Taylor circles of the orthic tetrastigm $II_1I_2I_3$.

(98) If the centres of these circles be denoted by

$$O_0, \ O_1, \ O_2, \ O_3$$

hen these four points form an orthic tetrastigm.

They are the incentre and the excentres of the complementary riangle A′B′C′.

(99) The six II lines of the orthic tetrastigm $II_1I_2I_3$ are the radical axes of the circles O_0, O_1, O_2, O_3 taken in pairs; and the four I points of the same tetrastigm are the radical centres of the circles O_0, O_1, O_2, O_3 taken in threes.

(100) The following are symmetrical trapeziums:

$$\begin{array}{lll} W_2V_3W_1V_1; & U_3W_1U_2W_2; & V_1U_2V_3U_3; \\ W_2V_3W'V'; & U'''W'U''W_2; & V'U''V_3U'''; \\ W''V'''W_1V'; & U_3W_1U''W''; & V'U''V'''U_3; \\ W''V'''W'V_1; & U'''W'U_2W''; & V_1U_2V'''U'''. \end{array}$$

Professor Fuhrmann gives the following property, but his proof is too long for insertion here :

The axis of homology of the triangles

$$\text{ABC and } X_1Y_2Z_3$$

is perpendicular to HI.

Of the last seventeen properties, (84), (85), (91) are given by W. H. Levy of Shalbourne in the *Lady's and Gentleman's Diary* for 1857, pp. 50-1, in his answer to a question proposed by him the previous year.

At the *Concours d'agrégation des sciences mathématiques* (Paris, 1873) the following question was proposed :

The points of contact of the excircles of a triangle ABC which are situated on the sides produced are joined, and a new triangle A'B'C' is formed. (1) *Find the angles of A'B'C'.* (2) *Prove that AA', BB', CC' are the altitudes of ABC.* (3) *Determine the centre and the radius of the circumcircle of A'B'C'.*

In the *Nouvelle Correspondance Mathématique*, I. 50-3 (1874), Professor Neuberg gives a geometrical solution of the question, in which (confining himself to triangle $X_1Y_2Z_3$) he proves (85), (86), (87), (91), (92), (93), (94) and one or two other properties. Professor Neuberg in the 6th edition of Casey's *Sequel to Euclid*, p. 278 (1892) and Professor Fuhrmann in his *Synthetische Beweise planimetrischer Sätze*, p. 89 (1890), give the first part of (97).

The seventeen properties were communicated to the Edinburgh Mathematical Society in 1889.

§ 6. EULER'S LINE.

*The circumcentre, the centroid, and the orthocentre of a triangle are collinear, and the distance between the first two is half the distance between the last two.**

FIGURE 61.

FIRST DEMONSTRATION.†

Let A′, B′ be the mid points of BC, CA, and let the perpendiculars to BC, CA at A′, B′ meet at O;
then O is the circumcentre.
Let the perpendiculars AX, BY meet at H;
then H is the orthocentre.

Join OH and let AA′ meet it at G. Join A′B′.

Because triangles HAB, OA′B′ have their sides respectively parallel to each other, they are similar;
therefore $HA : OA' = AB : A'B' = 2 : 1.$
Again triangles HAG, OA′G are similar;
therefore $HG : OG = HA : OA' = 2 : 1$
that is, AA′ cuts OH so that HG = twice OG.
Hence also the medians from B and C cut OH
so that HG = twice OG;
therefore G is the centroid, and H, G, O are collinear.

This is also a proof that the medians are concurrent.

SECOND DEMONSTRATION.

Let AA′ be the median from A, G the centroid, and O the circumcentre.

Join OA′, OG, and let AX the perpendicular from A to BC meet OG produced at H.

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* Proved by Euler in 1765. His proof will be found in *Novi Commentarii Academiae ... Petropolitanae*, XI. 114. An abstract of this paper of Euler's is printed in the *Proceedings of the Edinburgh Mathematical Society*, IV. 51-55 (1886).

† This method of proof is given in Carnot's *Géométrie de Position*, § 131 (1803). The second and third methods are imitations of it.

Then triangles HAG, OA′G are similar;

therefore $HG : OG = AG : A'G = 2 : 1$,

that is, AX cuts OG produced so that HG = twice OG.

Hence also the perpendiculars from B and C cut OG produced so that HG = twice OG;

therefore H is the orthocentre, and H, G, O are collinear.

This is also a proof that the perpendiculars to the sides from the vertices are concurrent.

Third Demonstration.

Let AX be the perpendicular, AA′ the median, from A to BC; and let H be the orthocentre, G the centroid.

Join HG, and let the perpendicular from A′ to BC meet HG produced at O.

Then triangles HAG, OA′G are similar;

therefore $HG : OG = AG : A'G = 2 : 1$,

that is, the perpendicular to BC from its mid point cuts HG produced so that HG = twice OG.

Hence also the perpendiculars to CA, AB from their mid points cut HG produced so that HG = twice OG;

therefore O is the circumcentre, and H, G, O are collinear.

This is also a proof that the perpendiculars to the sides from their mid points are concurrent.

Fourth Demonstration.*

Figure 62.

Let H be the orthocentre, determined by drawing AX, BY perpendicular to BC, CA; O the circumcentre, determined by drawing A′O, B′O perpendicular to BC, CA from their mid points A′, B′. Join HO and let it meet the median AA′ at G.

Bisect HA, HB at U, V, and GA, GH at P, Q;

join UV, PQ, A′B′.

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* This mode of proof assumes only the first book of Euclid's *Elements* and its immediate consequences.

Then $A'B'$ is parallel to AB and equal to $\frac{1}{2}AB$,
and UV is parallel to AB and equal to $\frac{1}{2}AB$;
therefore $A'B'$ is parallel to UV and equal to UV.
Because OA' and HU are both perpendicular to BC;
therefore OA' is parallel to HU.
Similarly OB' is parallel to HV.
Hence the triangles $OA'B'$, HUV are mutually equiangular, and, since $A'B' = UV$, congruent.
Therefore $$OA' = HU = \tfrac{1}{2}AH.$$

Again PQ is parallel to AH and equal to $\frac{1}{2}AH$;
therefore PQ is parallel to OA' and equal to OA'.
Hence the triangles $A'GO$, PGQ are congruent;
therefore $A'G = PG = \frac{1}{2}AG$;
therefore G is the centroid, and $OG = QG = \frac{1}{2}HG$.

The straight line HGO is frequently called *Euler's line.*

(1) *The twelve radii drawn from the incentre and the excentres of a triangle perpendicular to the sides of the triangle meet by threes in four points, and these four points are the circumcentres of the triangles* *

$$I_1I_2I_3, \quad II_3I_2, \quad I_3II_1, \quad I_2I_1I.$$

FIGURE 63.

The triads of concurrent radii are

$$\begin{array}{ll} I_1D_1,\ I_2E_2,\ I_3F_3 & ID,\ I_3E_3,\ I_2F_2 \\ I_3D_3,\ IE,\ I_1F_1 & I_2D_2,\ I_1E_1,\ IF \end{array}$$

and the theorem follows at once from the converse of the first part of § 5, (32).

A second proof of the concurrency of these four triads may be derived from Oppel's theorem in § 2 and the expressions in § 4, (5).

* The results (1)–(7) are given by T. S. Davies in the *Philosophical Magazine*, II. 26-34 (1827). The concurrency of the first triad at the circumcentre of triangle $I_1I_2I_3$, and the length of the radius, 2R, of that triangle were pointed out by Benjamin Bevan in Leybourn's *Mathematical Repository*, new series, I. 18 (pagination of questions), 143 (pagination of Part I.) in 1804. Compare the subscripts in the designations of the four I triangles with the subscripts of the radii which meet at their circumcentres.

A third proof may be got from a theorem of Steiner*:

If the three perpendiculars from the vertices of one triangle on the sides of another are concurrent, the three corresponding perpendiculars from the vertices of the latter on the sides of the former are also concurrent.

The following proof is due to Mr W. J. C. Miller †:

Because
$$\angle E_2I_2A = \tfrac{1}{2}CAB = \angle F_3I_3A$$
$$\angle F_3I_3B = \tfrac{1}{2}ABC = \angle D_1I_1B$$
$$\angle D_1I_1C = \tfrac{1}{2}BCA = \angle E_2I_2C;$$

therefore I_1D_1, I_2E_2, I_3F_3 will meet in a point O_0 such that
$$O_0I_1 = O_0I_2 = O_0I_3;$$

hence O_0 is the circumcentre of $I_1I_2I_3$.

Similarly for the other triads, which meet at the points
$$O_1,\ O_2,\ O_3.$$

DEF. Mr Lemoine has proposed ‡ to call triangles such as those of Steiner's theorem *orthologous*, and the points of concurrency of the perpendiculars *centres of orthology*.

Hence ABC is orthologous with each of the triangles
$$I_1I_2I_3,\quad II_3I_2,\quad I_3II_1,\quad I_2I_1I$$
and the respective centres of orthology are
$$I,\ O_0;\quad I_1,\ O_1;\quad I_2,\ O_2;\quad I_3,\ O_3.$$

(2) *The figures $O_0I_2O_1I_3$, $O_0I_3O_2I_1$, $O_0I_1O_3I_2$ are rhombi.*

For O_0I_2, O_1I_3 are perpendicular to AC

O_0I_3, O_1I_2 „ „ „ AB;

and
$$O_0I_2 = O_0I_3.$$

Hence O_0O_1, O_0O_2, O_0O_3

are respectively perpendicular to
$$I_2I_3,\quad I_3I_1,\quad I_1I_2.$$

* Crelle's *Journal*, II. 287 (1827), or Steiner's *Gesammelte Werke*, I. 157 (1881).

† *Lady's and Gentleman's Diary* for 1863, p. 54.

‡ *Journal de Mathématiques Spéciales*, 3rd series, III. 63 (1889), and the memoir *Sur les triangles orthologiques* read at the Limoges meeting (1890) of the *Association Française pour l'avancement des Sciences.*

(3) $$O_2O_3,\quad O_3O_1,\quad O_1O_2$$

are respectively parallel to

$$I_2I_3,\quad I_3I_1,\quad I_1I_2.$$

For O_2I_3, O_3I_2 are perpendicular to BC;
and they are equal, since the radii of the circumcircles of the four I triangles are equal;
therefore $O_2O_3I_2I_3$ is a parallelogram.

(4) The four O triangles are congruent to their respective four I triangles, and their corresponding sides are parallel.

(5) The points O_0, O_1, O_2, O_3 are the orthocentres of the four O triangles taken in order.

(6) *The figures $IO_2I_1O_3$, $IO_3I_2O_1$, $IO_1I_3O_2$ are rhombi.*

For IO_2, I_1O_3 are perpendicular to AC
IO_3, I_1O_2 " " " AB;

and $$IO_2 = IO_3,$$

since the radii of the circumcircles of the four I triangles are equal.

(7) The points I, I_1, I_2, I_3 are the circumcentres of the four O triangles taken in order.

(8) $$IO_0,\quad I_1O_1,\quad I_2O_2,\quad I_3O_3$$

are the Euler's lines of the four I triangles, and of the four O triangles, and the circumcentre O of ABC is the mid point of each of them.

(9) By referring to the Section* on the nine-point circle, it will be seen that the circumcircle of ABC is the nine-point circle of the eight I and O triangles, and that the radii of the circumcircles of these eight triangles are each 2R.

It will also be seen that the circumcircle of ABC bisects each of the six straight lines

$$II_1,\quad II_2,\quad II_3,\quad I_2I_3,\quad I_3I_1,\quad I_1I_2$$

* *Proceedings of the Edinburgh Mathematical Society*, XI. 19-57 (1893).

at $U\ ,\ V\ ,\ W\ ,\ U'\ ,\ V'\ ,\ W'$;

and that $UU'\ ,\ VV'\ ,\ WW'$

are the diameters of the circumcircle ABC perpendicular to

$$BC\ ,\ CA\ ,\ AB.$$

(10) $U'V'W'\ ,\ U'VW\ ,\ UV'W\ ,\ UVW'$

are the complementary triangles of the four I triangles taken in order.

(11) $O_0O_1\ ,\ O_0O_2\ ,\ O_0O_3\ ,\ O_2O_3\ ,\ O_3O_1\ ,\ O_1O_2$

pass respectively through the points

$$U'\ ,\ V'\ ,\ W'\ ,\ U\ ,\ V\ ,\ W.$$

(12) The following pairs of straight lines intersect on the circumcircle of ABC:

$$O_0O_1\ ,\ O_2O_3\ ;\quad O_0O_2\ ,\ O_3O_1\ ;\quad O_0O_3\ ,\ O_1O_2$$

at $U_1\ ,\ V_1\ ,\ W_1$.

(13) Triangle $U_1V_1W_1$ bears to $O_0O_1O_3$ exactly the same relations that ABC does to $I_1I_2I_3$.

FIGURE 64.

(14) Of the four I triangles taken in order let

$$G_0\ ,\ G_1\ ,\ G_2\ ,\ G_3$$

be the centroids; then the concurrency of

I_1U', I_2V', I_3W'	determines	G_0
$I\,U'$, I_3V, I_2W	,,	G_1
I_3U, $I\,V'$, I_1W	,,	G_2
I_2U, I_1V, $I\,W'$	,,	G_3.

(15)

G_0	lies on	$I\,O_0$	and	$I\,G_0 = 2O_0G_0$
G_1	,, ,,	I_1O_1	,,	$I_1G_1 = 2O_1G_1$
G_2	,, ,,	I_2O_2	,,	$I_2G_2 = 2O_2G_2$
G_3	,, ,,	I_3O_3	,,	$I_3G_3 = 2O_3G_3$.

(16) Through O pass*

$IG_0\ ,\ I_1G_1\ ,\ I_2G_2\ ,\ I_3G_3$ and

$$OI = 3OG_0\ ,\ OI_1 = 3OG_1\ ,\ OI_2 = 3OG_2\ ,\ OI_3 = 3OG_3.$$

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* Thomas Weddle in the *Lady's and Gentleman's Diary* for 1849, p. 76.

(17) If through G_0 parallels be drawn to

$$O_0U', \quad O_0V', \quad O_0W'$$

these parallels will meet

$$I\,U', \quad I\,V', \quad I\,W'$$

at

$$G_1, \quad G_2, \quad G_3 .$$

(18) $G_1G_2G_3$ is directly similar to $U'V'W'$ and $I_1I_2I_3$, the ratio of similitude in the first case being $2 : 3$, and in the second $1 : 3$.

(19) I, U, V, W; O_0, U', V', W'; G_0, G_1, G_2, G_3 form orthic tetrastigms.

FIGURE 63.

(20) The areas of the six rhombi *

$$O_0I_2O_1I_3, \quad O_0I_3O_2I_1, \quad O_0I_1O_3I_2$$
$$I\,O_2I_1O_3, \quad I\,O_3I_2O_1, \quad I\,O_1I_3O_2$$

are

$$2Ra, \quad 2Rb, \quad 2Rc .$$

(21) The areas of the three parallelograms †

$$I_2I_3O_2O_3, \quad I_3I_1O_3O_1, \quad I_1I_2O_1O_2$$

are

$$2R(b+c), \quad 2R(c+a), \quad 2R(a+b) .$$

(22) The figure $I_1O_3I_2O_1I_3O_2$ is an equilateral hexagon ‡; its opposite sides are parallel ‡, and equal to the diameter of the circumcircle ‡ of ABC; its angles are the supplements § of the angles of ABC; and its area § is equal to the sum of the areas of $I_1I_2I_3$ and $O_1O_2O_3$, that is equal to $4Rs$.

* The last three are given by Rev. William Mason of Normanton in the *Lady's and Gentleman's Diary* for 1863, p. 53.

† Mr S. Constable in the *Educational Times*, XXXI. 113 (1878).

‡ T. S. Davies in the *Philosophical Magazine*, II. 32 (1827).

§ Rev. William Mason in the *Lady's and Gentleman's Diary* for 1863, p. 54.

§ 7. Relations among Radii.

*The sum of the radii of the three excircles diminished by the radius of the incircle is double the diameter of the circumcircle.**

Figure 64.

Let O be the circumcentre, ID, I_1D_1, I_2D_2, I_3D_3 the radii of the incircle and the three excircles perpendicular to BC.

From O draw OA′ perpendicular to BC, and let OA′ meet the circumcircle below BC at U and above it at U′.

Because OA′ is perpendicular to BC,
therefore A′ is the mid point of BC, and U the mid point of arc BUC.
But since AI_1 bisects $\angle$ BAC,
therefore it bisects arc BUC, that is, AI_1 passes through U.
Since UU′ is a diameter, and $\angle UAI_3$ is right,
therefore I_2I_3 passes through U′.
Now because ID, UA′, I_1D_1 are parallel, and $DA' = D_1A'$,

therefore $$2UA' = I_1D_1 - ID\ ;$$

and because I_2D_2, $U'A'$, I_3D_3 are parallel, and $D_2A' = D_3A'$,

therefore $$2U'A' = I_2D_2 + I_3D_3\,.$$

Hence $$2(UA' + U'A') = I_1D_1 + I_2D_2 + I_3D_3 - ID\ ;$$

that is $$4R = r_1 + r_2 + r_3 - r\,.$$

(1) *The sum of the distances of the circumcentre from the sides of a triangle is equal to the sum of the radii of the incircle and the circumcircle; and the sum of the distances of the orthocentre from the vertices is equal to the sum of the diameters of the incircle and the circumcircle.*

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* Feuerbach in his *Eigenschaften...des...Dreiecks*, § 5 (1822), proves it algebraically. The proof in the text is that of T. S. Davies in the *Ladies' Diary* for 1835, p. 55.

For $$OA' = OU - A'U$$
$$= R - \tfrac{1}{2}(r_1 - r).$$

Similarly $$OB' = R - \tfrac{1}{2}(r_2 - r);$$
$$OC' = R - \tfrac{1}{2}(r_3 - r);$$

therefore $$OA' + OB' + OC' = 3R - \tfrac{1}{2}(r_1 + r_2 + r_3 - 3r)$$
$$= 3R - \tfrac{1}{2}(4R + r - 3r)$$
$$= 3R - (2R - r)$$
$$= R + r.$$

Again $$HA + HB + HC = 2OA' + 2OB' + 2OC',$$
$$= 2(R + r).$$

If one of the angles of the triangle be obtuse, the circumcentre will fall outside the triangle, and its distance from the side opposite the obtuse angle must then be considered negative. Also if the circumcentre fall outside the triangle, so will the orthocentre. In that case the distance of the orthocentre from the vertex of the obtuse angle must be considered negative.

These two properties, as well as the remarks at the end of the proof, are given by Carnot in his *Géométrie de Position*, § 137 (1803).

The following is Carnot's mode of proving the first property.

FIGURE 65.

The quadrilaterals AB'OC', BA'OC', CB'OA' are inscriptible in circles; therefore

$$AO \cdot B'C' = AB' \cdot C'O + AC' \cdot B'O$$
$$BO \cdot C'A' = BC' \cdot A'O + BA' \cdot C'O$$
$$CO \cdot A'B' = CA' \cdot B'O + CB' \cdot A'O.$$

Adding these equations, and noting that $AO = BO = CO = R$, that $B'C' + C'A' + A'B' = s$, and that $AB' = CB'$, $AC' = BC'$, $BA = CA'$, we have

$$sR = s(A'O + B'O + C'O) - \tfrac{1}{2}(A'O \cdot BC + B'O \cdot CA + C'O \cdot AB)$$
$$= s(A'O + B'O + C'O) - \triangle$$
$$= s(A'O + B'O + C'O) - sr;$$

therefore $\mathrm{R} = \mathrm{A'O} + \mathrm{B'O} + \mathrm{C'O} - r,$

or $\mathrm{R} + r = \mathrm{A'O} + \mathrm{B'O} + \mathrm{C'O}.$

(2) *The relation* $4R = r_1 + r_2 + r_3 - r$ *has been employed to establish* $R + r = OA' + OB' + OC'$; *but the method of procedure may be reversed.*

FIGURE 64.

For $\mathrm{OA'} = \mathrm{OU} - \mathrm{A'U} = \mathrm{R} - \frac{1}{2}(r_1 - r).$

Similarly $\mathrm{OB'} = \mathrm{R} - \frac{1}{2}(r_2 - r)$

$\mathrm{OC'} = \mathrm{R} - \frac{1}{2}(r_3 - r)$;

therefore $\mathrm{OA'} + \mathrm{OB'} + \mathrm{OC'} = 3\mathrm{R} - \frac{1}{2}(r_1 + r_2 + r_3 - 3r)$

that is $\mathrm{R} + r = 3\mathrm{R} - \frac{1}{2}(r_1 + r_2 + r_3 - 3r)$;

whence $4\mathrm{R} = r_1 + r_2 + r_3 - r.$

(3) $A'U + B'V + C'W = 2R - r$

*and** $A'U' + B'V' + C'W' = 4R + r.$

These results follow from subtracting $\mathrm{A'O} + \mathrm{B'O} + \mathrm{C'O}$ from 3R, and adding $\mathrm{A'O} + \mathrm{B'O} + \mathrm{C'O}$ to 3R.

For another proof of the first of them see *Mathematical Questions from the Educational Times*, XVII. 47 (1872).

(4) *The following relations subsist between the distances of the circumcentre from the sides of a triangle and the radii of the circumcircle and the excircles :* †

$$-OA' + OB' + OC' = -R + r_1$$
$$OA' - OB' + OC' = -R + r_2$$
$$OA' + OB' - OC' = -R + r_3.$$

FIGURE 66.

From U draw UT perpendicular to AB,

and from O draw OR perpendicular to UT.

* Hind's *Trigonometry*, 4th ed., p. 309 (1841).

† Mr Bernh. Möllmann in Grunert's *Archiv*, XVII. 379 (1851).

It may be proved that $AT = \frac{1}{2}(AB + AC)$

and $BT = \frac{1}{2}(AB - AC)$;

therefore $AB' = \frac{1}{2}AC = \frac{1}{2}(AT - BT)$

$= C'T = OR.$

Hence the right-angled triangles AB'O, ORU are congruent

and $OB' = UR$;

therefore $OB' + OC' = UT$

$= \frac{1}{2}(IF + I_1F_1)$;

therefore $2(OB' + OC') = r + r_1.$

Now $OA' + OB' + OC' = R + r$;

therefore $-OA' + OB' + OC' = -R + r_1.$

(5) *The following is another proof* of the relation*

$$OA' + OB' + OC' = R + r.$$

FIGURE 67.

Through I the incentre draw a parallel to AC meeting OB' in K; through K draw a parallel to IC meeting OA' in L and BC in N. From N draw NM perpendicular to AC and meeting OA' in M.

Since KN is parallel to the bisector of $\angle ACB$,

therefore $CN = IK = EB' = CB' - CE$

$$= \frac{AC}{2} - \frac{BC + CA - AB}{2}$$

$$= \frac{AB - BC}{2};$$

therefore $A'N = A'C + CN = \frac{AB}{2} = C'B.$

Again, since MN is perpendicular to AC,

therefore $\angle MNA' = 90° - \angle ACB$

$= 90° - \angle C'OB = \angle OBC'$;

* Mr Lemoine in *Journal de Mathématiques Élémentaires*, 2nd series, IV. 217-8 (1885). (6) also is his.

therefore the right-angled triangles MNA′, OBC′ are congruent,

and $$A'M = OC', \quad MN = OB = R.$$

Since
$$\begin{aligned}\angle MNL &= \angle MNA' + \angle A'NL \\ &= 90^\circ - C + \frac{C}{2} \\ &= 90^\circ - \frac{C}{2} \\ &= \angle MLN;\end{aligned}$$

therefore triangle MLN is isosceles;

therefore " OLK " " .

Hence
$$\begin{aligned}OA' + OB' + OC' &= OA' + OK + KB' + A'M \\ &= OA' + OL + IE + A'M \\ &= LM + IE \\ &= MN + IE \\ &= R + r.\end{aligned}$$

(6) If r_1 denote the radius of the first excircle, it may be shown by an analogous proof that

$$OB' + OC' - OA' = r_1 - R.$$

Hence the theorem:

If in a triangle the radius of an excircle be equal to the radius of the circumcircle, one of the three distances of the orthocentre from the vertices is equal to the sum of the other two, and conversely.

(7) *If D, D' be the projections of A' on OB, OC,*
E, E' " " " " B' " OC, OA,
F, F' " " " " C' " OA, OB

*then** $$\sqrt{\frac{DD'}{a}} + \sqrt{\frac{EE'}{b}} + \sqrt{\frac{FF'}{c}} = \frac{R+r}{R}.$$

FIGURE 68.

Since triangle OBC is isosceles, DD′ is parallel to BC,

and $$\frac{DD'}{a} = \frac{OD}{R}.$$

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* Mr J. Soméritis of Chalcis in Vuibert's *Journal de Mathématiques Élémentaires*, XVI. 128 (1892). The solution in the text is that given on p. 141.

From the right-angled triangles OA′B, ODA′

$$OD = \frac{OA'^2}{R};$$

therefore $\dfrac{DD'}{a} = \dfrac{OA'^2}{R^2}$ and $\sqrt{\dfrac{DD'}{a}} = \dfrac{OA'}{R}$.

Similarly $\sqrt{\dfrac{EE'}{b}} = \dfrac{OB'}{R}$ and $\sqrt{\dfrac{FF'}{c}} = \dfrac{OC'}{R}$;

therefore $$\sqrt{\frac{DD'}{a}} + \sqrt{\frac{EE'}{b}} + \sqrt{\frac{FF'}{c}} = \frac{OA' + OB' + OC'}{R}$$

$$= \frac{R + r}{R}.$$

(8) *The potency (or power) of the incentre of a triangle with respect to the circumcircle is equal to twice the rectangle under the radii of the incircle and the circumcircle.**

FIGURE 69.

In ABC let O be the circumcentre, I the incentre.

Join OI.

Through O draw U′U the diameter of the circumcircle perpendicular to BC. Then U is the mid point of the arc BUC, and AU will pass through I.

Join CU, CU′, CI, AO, and from I draw IE the radius of the incircle perpendicular to AC.

Then $$\angle UIC = \angle IAC + \angle ICA,$$
$$= \tfrac{1}{2}(A + C);$$

and $$\angle UCI = \angle BCI + \angle BCU,$$
$$= \angle BCI + \angle BAU,$$
$$= \tfrac{1}{2}(A + C);$$

therefore $$CU = IU.$$

* William Chapple in *Miscellanea Curiosa Mathematica*, I. 123 (1746). Euler gave the property in an inconvenient form about twenty years later. A tolerably full history of Chapple's theorem and its developments during the 18th century will be found in the *Proceedings of the Edinburgh Mathematical Society*, V. 62-78 (1887).

Again, the right-angled triangles AEI, U'CU are similar;

therefore $AI : IE = U'U : UC$;

therefore $AI \cdot UC = U'U \cdot IE$,

that is $AI \cdot IU = 2Rr$.

Lastly from the isosceles triangle OAU

$$OA^2 - OI^2 = AI \cdot IU,$$

that is $R^2 - OI^2 = 2Rr$.

(9) If OI be denoted by d, then

$$R^2 - d^2 = 2Rr,$$

or $$\frac{1}{R+d} + \frac{1}{R-d} = \frac{1}{r}.$$

(10) *The potency (or power) of an excentre of a triangle with respect to the circumcircle is equal to twice the rectangle under the radii of the excircle and the circumcircle.**

FIGURE 70.

In ABC let O be the circumcentre, I_1 an excentre.

Join OI_1.

Through O draw U'U the diameter of the circumcircle perpendicular to BC. Then U is the mid point of the arc BUC, and AU will pass through I_1,

Join CU, CU', CI_1, AO, and from I_1 draw I_1E_1 the radius of the excircle perpendicular to AC.

Then $\angle UI_1C = 180° - (\angle I_1AC + \angle I_1CA)$,

$= 90° - \frac{1}{2}(A+C)$;

and $\angle UCI_1 = \angle BCI_1 - \angle BCU$,

$= \angle BCI_1 - \angle BAU$,

$= 90° - \frac{1}{2}(A+C)$;

therefore $CU = I_1U$.

Again, the right-angled triangles AE_1I_1, U'CU are similar;

therefore $AI_1 : I_1E_1 = U'U : UC$;

therefore $AI_1 \cdot UC = U'U \cdot I_1E_1$,

that is $AI_1 \cdot I_1U = 2Rr_1$.

* John Landen in *Lucubrationes Mathematicae*, pp. 1-6 (1755).

Lastly from the isosceles triangle OAU,

$$OI_1^2 - OA^2 = AI_1 \cdot I_1U,$$

that is $$OI_1^2 - \ R^2 \ = 2Rr_1.$$

Hence also $$OI_2^2 - \ R^2 \ = 2Rr_2,$$

and $$OI_3^2 - \ R^2 \ = 2Rr_3.$$

(11) If OI_1, OI_2, OI_3 be denoted by d_1, d_2, d_3 then

$$d_1^2 - R^2 = 2Rr_1,\ d_2^2 - R^2 = 2Rr_2,\ d_3^2 - R^2 = 2Rr_3\,;$$

or
$$\frac{1}{R+d_1} + \frac{1}{R-d_1} = -\frac{1}{r_1}$$
$$\frac{1}{R+d_2} + \frac{1}{R-d_2} = -\frac{1}{r_2}$$
$$\frac{1}{R+d_3} + \frac{1}{R-d_3} = -\frac{1}{r_3}.$$

(12) The potency of I with respect to the circumcircle is *

$$\frac{abc}{a+b+c}.$$

For $$2Rr = \frac{abc}{2\triangle} \cdot \frac{2\triangle}{a+b+c}.$$

(13) The potency of I_1 with respect to the circumcircle is

$$\frac{abc}{-a+b+c}.$$

For $$2Rr_1 = \frac{abc}{2\triangle} \cdot \frac{2\triangle}{-a+b+c}.$$

(14) If the first excircle cut the circumcircle at S, and I_1S be produced to intersect the circumcircle at T, then $I_1T = 2R$.

For $I_1S \cdot I_1T$ = potency of I_1 with respect to circumcircle,

$$= OI_1^2 - R^2 = 2Rr_1\,;$$

and $I_1S = r_1$.

---

* C. J. Matthes, *Commentatio de Proprietatibus Quinque Circulorum*, p. 41 (1831).

(15) If IO be produced to meet the circumcircle in M, N, and the incircle in P, Q (the order of the letters is MPOIQN), then *

$$MP \cdot NQ = r^2$$
$$MQ \cdot NP = 4Rr + r^2.$$

For $\qquad MP = (R - r) + OI, \quad NQ = (R - r) - OI$

and $\qquad MQ = (R + r) + OI, \quad NP = (R + r) - OI.$

(16) If I_1O be produced to meet the circumcircle in M, N, and the first excircle in P, Q (the order of the letters is $MOQNI_1P$), then

$$MP \cdot NQ = r_1^2$$
$$MQ \cdot NP = 4Rr_1 - r_1^2.$$

(17) $\qquad IM \cdot IN = 2Rr$

$$OP \cdot OQ = -R^2 + 2Rr + r^2.$$

(18) $\qquad I_1M \cdot I_1N = 2Rr_1$

$$OP \cdot OQ = R^2 + 2Rr_1 - r_1^2.$$

(19) The product of the potencies † of P and Q with respect to the circumcircle

$$MP \cdot NP \times MQ \cdot NQ = r^3(4R + r).$$

The product of the potencies of M and N with respect to the incircle

$$MP \cdot MQ \times NP \cdot NQ = r^3(4R + r).$$

(20) The product of the potencies of P and Q with respect to the circumcircle

$$MP \cdot NP \times MQ \cdot NQ = r_1^3(4R - r_1).$$

The product of the potencies of M and N with respect to the first excircle

$$MP \cdot MQ \times NP \cdot NQ = r_1^3(4R - r_1).$$

---

* The first part is given by Mr Nèorouzian in the *Nouvelles Annales*, IX. 216-7 (1850); the second part occurs in *Exercices de Géométrie*, by F.I.C., 2nd ed., p. 506 (1882).

† The first part is given in *Nouvelles Annales*, XVII. 358, 447-8 (1858), and attributed to Grunert.

(21) The radius of the circumcircle is never less than the diameter of the incircle.*

For OI^2 is positive;

therefore $R - 2r$ cannot be negative.

(22) When the radius of the circumcircle is equal to the diameter of the incircle, the circumcentre and the incentre coincide, and the triangle is equilateral.

(23) When the straight line joining the incentre and the circumcentre passes through one of the vertices, the triangle is isosceles.

(24) Since the value of OI^2 is independent of the sides of the triangle ABC, if two circles whose radii are R and r be so situated that the square of the distance between their centres equals $R(R - 2r)$, then any number of triangles may be drawn, each of which shall be inscribed in the larger circle, and circumscribed about the smaller†; and if the two circles be so situated that the square of the distance between their centres is not equal to $R(R - 2r)$, then no triangle can be inscribed and circumscribed.

(25) Since the value of OI_1^2 is independent of the sides of the triangle, a corresponding statement may be made regarding two circles whose radii are R and r_1.

(26) If one side of a triangle inscribed in and circumscribed about two given circles be given, the other two sides may be found.

(27) Of the innumerable triangles that may be inscribed in and circumscribed about two given circles, two will be isosceles; and the common diameter of the two circles will pass through their vertices and cut their bases at right angles. That isosceles triangle which has the least base and the greatest altitude will be the greatest, and the other isosceles triangle will be the least of all the triangles that can be inscribed and circumscribed.

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* Theorems (21)–(24), (26), (27) are given by Chapple; (28) part of which is given by Chapple, is due to Dr Otto Böklen. See Grunert's *Archiv*, XXXVIII. 143 (1862).

† A detailed proof of this statement, if such should be considered necessary, is given by Dr W. H. Besant in the *Quarterly Journal of Mathematics*, XII. 276 (1873).

(28) In connection with these innumerable triangles a large number of constant magnitudes may be found. A few are here enumerated.

(*a*) The sum of the perpendiculars from the circumcentre to the sides is constant.

(*b*) The sum of the distances of the orthocentre from the vertices is constant.

(*c*) The sum of the radii of the excircles is constant.

(*d*) The sum of the reciprocals of the radii of the excircles is constant.

(*e*) The ratio of the product of the sides to the sum of the sides is constant.

(*f*) The ratio of the area to the perimeter is constant.

The proofs of these statements are

(*a*) $$OA' + OB' + OC' = R + r$$

(*b*) $$\tfrac{1}{2}(HA + HB + HC) = R + r$$

(*c*) $$r_1 + r_2 + r_3 = R + r$$

(*d*) $$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$$

(*e*) $$\frac{abc}{a+b+c} = 2Rr$$

(*f*) $$\frac{\Delta}{s} = r \quad .$$

(29) The sum of the squares of the distances of the circumcentre from the incentre and the excentres is equal to three times the square of the diameter of the circumcircle.*

For
$$\begin{aligned} \Sigma(OI^2) &= 4R^2 + 2R(r_1 + r_2 + r_3 - r) \\ &= 4R^2 + 2R \cdot 4R \\ &= 12R^2. \end{aligned}$$

* Feuerbach, *Eigenschaften...des...Dreiecks*, § 50 (1822).

(30) The preceding theorem may be derived from the following*:

The sum of the squares of the tangents drawn from the centres of the four circles of contact of a triangle to any circle which passes through the circumcentre is equal to three times the square of the diameter of the circumcircle.

FIGURE 71.

Let Q be the centre of a circle passing through O the circumcentre.

Draw the diameter of the circumcircle UU′ perpendicular to BC and bisecting II_1 at U and I_2I_3 at U′.

Join Q with O, I, I_1, I_2, I_3, U, U′, and draw CU, CU′.

If the four tangents be denoted by t, t_1, t_2, t_3,

$$
\begin{aligned}
\text{then } t^2+t_1^2+t_2^2+t_3^2 &= (QI^2-QO^2)+(QI_1^2-QO^2) \\
&\quad+(QI_2^2-QO^2)+(QI_3^2-QO^2) \\
&= (QI^2+QI_1^2)+(QI_2^2+QI_3^2)-4QO^2 \\
&= 2(QU^2+UI^2)+2(QU'^2+U'I_2^2)-4QO^2 \\
&= 2(QU^2+UC^2)+2(QU'^2+U'C^2)-4QO^2 \\
&= 2(QU^2+QU'^2)+2(UC^2+U'C^2)-4QO^2 \\
&= 4(QO^2+OU^2)+2U'U^2-4QO^2 \\
&= 4OU^2+8OU^2 \\
&= 12R^2.
\end{aligned}
$$

When QO becomes zero, or the circle with centre Q vanishes to a point,

$$t^2+t_1^2+t_2^2+t_3^2 = OI^2+OI_1^2+OI_2^2+OI_3^2.$$

(31) Since $-\frac{1}{r}+\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=0,$

therefore $-\frac{1}{2Rr}+\frac{1}{2Rr_1}+\frac{1}{2Rr_2}+\frac{1}{2Rr_3}=0;$

therefore $\frac{1}{d^2-R^2}+\frac{1}{d_1^2-R^2}+\frac{1}{d_2^2-R^2}+\frac{1}{d_3^2-R^2}=0;$

therefore $\frac{1}{12(R^2-d^2)}+\frac{1}{12(R^2-d_1^2)}+\frac{1}{12(R^2-d_2^2)}+\frac{1}{12(R^2-d_3^2)}=0.$

* Philip Beecroft in the *Lady's and Gentleman's Diary* for 1845, p. 63.

But
$$12(R^2-d^2)=d_1^2+d_2^2+d_3^2-11d^2$$
$$12(R^2-d_1^2)=d^2+d_2^2+d_3^2-11d_1^2$$
$$12(R^2-d_2^2)=d_1^2+d^2+d_3^2-11d_2^2$$
$$12(R^2-d_3^2)=d_1^2+d_2^2+d^2-11d_3^2;$$

hence
$$\frac{1}{d_1^2+d_2^2+d_3^2-11d^2}+\frac{1}{d^2+d_2^2+d_3^2-11d_1^2}+$$
$$\frac{1}{d_1^2+d^2+d_3^2-11d_2^2}+\frac{1}{d_1^2+d_2^2+d^2-11d_3^2}=0,$$

the equation* by which the four distances d, d_1, d_2, d_3 are connected together.

(32)
$$\left.\begin{aligned}
-\frac{1}{d^2-R^2}+\frac{1}{d_1^2-R^2}+\frac{1}{d_2^2-R^2}+\frac{1}{d_3^2-R^2}&=\frac{1}{Rr}\\
-\frac{1}{d^2-R^2}+\frac{1}{d_1^2-R^2}-\frac{1}{d_2^2-R^2}-\frac{1}{d_3^2-R^2}&=\frac{1}{Rr_1}\\
-\frac{1}{d^2-R^2}-\frac{1}{d_1^2-R^2}+\frac{1}{d_2^2-R^2}-\frac{1}{d_3^2-R^2}&=\frac{1}{Rr_2}\\
-\frac{1}{d^2-R^2}-\frac{1}{d_1^2-R^2}-\frac{1}{d_2^2-R^2}+\frac{1}{d_3^2-R^2}&=\frac{1}{Rr}
\end{aligned}\right\}$$

A large number of formulae expressive of the relations between r, r_1, r_2, r_3, R, and a, b, c, h_1, h_2, h_3, α, β, γ, etc., will be found in subsequent Sections.

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* Mr Franz Unferdinger in Grunert's *Archiv*, XXXIII. 428 (1859).

§ 8. AREA.

The area of a triangle is a mean proportional between the rectangle contained by the semiperimeter and its excess above any one side and the rectangle contained by its excesses above the other two sides.

FIRST DEMONSTRATION.

FIGURE 72.

Let ABC be the given triangle, and let each of its sides be given: to find the area.

Inscribe in the triangle the circle DEF whose centre is I, and join I with the points A, B, C, D, E, F.

Then the rectangle $BC \cdot ID =$ twice triangle BCI,

the rectangle $CA \cdot IE =$ twice triangle CAI,

and the rectangle $AB \cdot IF =$ twice triangle ABI;

hence the rectangle under the perimeter of triangle ABC and ID, the radius of the circle DFE = twice triangle ABC.

Produce BC, and make CD_2 equal to AE; then BD_2 is the semiperimeter, and the rectangle $BD_2 \cdot ID =$ triangle ABC.
But the rectangle $BD_2 \cdot ID$ is a side of the solid contained by BD_2 and the square of ID; therefore the area of the triangle will be a side of the solid contained by BD_2 and the square of ID.

Draw IL perpendicular to IB, CL perpendicular to CB, and join BL.

Since each of the angles BIL, BCL is right,
the points B, I, C, L are concyclic;
therefore the angles BIC, BLC are equal to two right angles.
But the angles BIC, AIE are equal to two right angles,
because AI, BI, CI bisect the angles at the point I;
therefore angle AIE = angle BLC,
and triangle AIE is similar to triangle BLC.

Hence $BC : LC = AE : IE,$
$= CD_2 : ID;$

therefore $BC : CD_2 = LC : ID$, by alternation,
$= CK : DK;$

and $BD_2 : CD_2 = CD : DK$, by composition.

Consequently $BD_2^2 : BD_2 \cdot CD_2 = CD \cdot BD : BD \cdot DK,$

$= CD \cdot BD : ID^2$;

therefore $BD_2^2 \cdot I D^2 = BD_2 \cdot CD_2 \cdot BD \cdot CD.$

Now each of the lines BD_2, CD_2, BD, CD is given ;
for BD_2 is the semiperimeter, CD_2 the excess of the semiperimeter above BC, BD the excess of the semiperimeter above AC, and CD the excess of the semiperimeter above AB.
The area of the triangle therefore is given.

[NUMERICAL ILLUSTRATION.]

Let AB consist of 13 parts, BC of 14, CA of 15.
Add the three together; the result is 42, of which the half is 21.
Subtract 13; there remain 8: 14, there remain 7 : 15, there remain 6.
21, 8, 7, 6 into one another produce 7056, the square root of which is 84.

The area of the triangle is 84.

This useful theorem occurs in a treatise "On the Dioptra" (περὶ διόπτρας) which many mathematical historians attribute to Heron of Alexandria (about 120 B.C.). See Cantor's *Vorlesungen über Geschichte der Mathematik*, I. 322-6 (1880). Mr Maximilien Marie, however (*Histoire des Sciences Mathématiques et Physiques*, I. 177-190), thinks the theorem cannot belong to so early a period, and ascribes it to Heron of Constantinople. The theorem was known to the Hindu mathematician Brahmegupta (born 598 A.D.) and to the Arabs. A good deal of historical information regarding it will be found in Chasles' *Aperçu Historique*, Note XII.

I have translated the demonstration in the text from Hultsch's *Heronis Alexandrini Geometricorum et Stereometricorum Reliquiae*, pp. 235-7 (1864), but I have not transliterated the notation.

SECOND DEMONSTRATION.

FIGURE 36.

Let ABC be a triangle, AX the perpendicular* from A to BC.

Then $AB^2 = BC^2 + CA^2 - 2BC \cdot CX$

that is $c^2 = a^2 + b^2 - 2a \cdot CX$;

therefore $$CX = \frac{a^2 + b^2 - c^2}{2a}.$$

* Whatever be the shape of the triangle one of the perpendiculars will always fall inside the triangle. Let that perpendicular be AX.

Now $$AX^2 = AC^2 - CX^2$$
$$= b^2 - \left(\frac{a^2+b^2-c^2}{2a}\right)^2$$
$$= \frac{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}{4a^2}$$
$$= \frac{2s \cdot 2s_1 \cdot 2s_2 \cdot 2s_3}{4a^2};$$

therefore $$AX = \frac{2}{a}\sqrt{ss_1s_2s_3}.$$

Hence $$\triangle = \tfrac{1}{2}BC \cdot AX$$
$$= \frac{a}{2} \cdot \frac{2}{a}\sqrt{ss_1s_2s_3}$$
$$= \sqrt{ss_1s_2s_3}.$$

THIRD DEMONSTRATION.

FIGURE 28.

Because triangles AFI, AF_1I_1 are similar,

therefore $AF : IF = AF_1 : I_1F_1$;

therefore $AF_1 \cdot AF : AF_1 \cdot IF = AF_1 \cdot IF : I_1F_1 \cdot IF$.

Because triangles IBF, BI_1F_1 are similar,

therefore $BF : IF = I_1F_1 : BF_1$;

therefore $IF \cdot I_1F_1 = BF \cdot BF_1$.

Hence $AF_1 \cdot AF : AF_1 \cdot IF = AF_1 \cdot IF : BF \cdot BF_1$;

therefore $ss_1 : \triangle = \triangle : s_2s_3$.

(1) $$\triangle = \tfrac{1}{4}\sqrt{2(b^2c^2+c^2a^2+a^2b^2)-(a^4+b^4+c^4)}.$$

This expression is convenient when a, b, c are irrational quantities.

(2) *The following method will enable us to discover the expression for the area of a triangle, if it is known that the square of its area is an integral function of its sides.**

* Terquem in *Nouvelles Annales*, III. 219-220 (1844). The method is also applied by Terquem to find the expression for the area of a cyclic quadrilateral, and it had previously been applied by P. L. Cirodde in *Nouvelles Annales*, I. 117 (1842), to find the volume of a spherical segment when it is known that the volume is a function of the third degree of its height.

Let Δ denote the area of the triangle, a, b, c its sides.

Then Δ^2 is a symmetrical function of the sides of the fourth degree. If one side becomes equal to the sum of the two others, the area vanishes;
therefore Δ^2 contains the three factors $-a+b+c$, $a-b+c$, $a+b-c$. The fourth factor must therefore be of the form $m(a+b+c)$, where m is a constant number;
therefore $\Delta^2 = m(a+b+c)(-a+b+c)(a-b+c)(a+b-c)$.

To determine the value of m, suppose the three sides of the triangle to be equal;
then $$\Delta^2 = 3ma^4.$$
But the square of the area of an equilateral triangle,
whose side is a, $$= \frac{3}{16}a^4;$$
therefore $$m = \frac{1}{16}.$$

Formulae for the Areas of Certain Triangles.

See the notation, pp. 7–11.

Triangles connected with the Centroid.

$$A'B'C' = AC'B' = C'BA' = B'A'C = \tfrac{1}{4}\Delta \tag{1}$$

If R, S, T be the projections of G on the sides

$$GST = \frac{4}{9}\frac{\Delta^3}{b^2c^2} \qquad GTR = \frac{4}{9}\frac{\Delta^3}{c^2a^2} \qquad GRS = \frac{4}{9}\frac{\Delta^3}{a^2b^2} \tag{2}$$

$$RST = \frac{4}{9}\cdot\frac{a^2+b^2+c^2}{a^2b^2c^2}\Delta^3 \tag{3}$$

Triangles connected with the Circumcentre.

$$\left.\begin{aligned} OBC &= \tfrac{1}{2}ak_1 = \frac{a^2(-a^2+b^2+c^2)}{16\Delta} \\ OCA &= \tfrac{1}{2}bk_2 = \frac{b^2(a^2-b^2+c^2)}{16\Delta} \\ OAB &= \tfrac{1}{2}ck_3 = \frac{c^2(a^2+b^2-c^2)}{16\Delta} \end{aligned}\right\} \tag{4}$$

Since O is the orthocentre of A′B′C′,

therefore OC′B′, C′OA′, B′A′O

stand in the same relation to A′B′C′ as

HCB, CHA, BAH

stand to ABC.

Hence expressions for the areas of these triangles may be derived from (27).

Triangles connected with the Incentre and Excentres.

$$\frac{DEF}{r} = \frac{D_1E_1F_1}{r_1} = \frac{D_2E_2F_2}{r_2} = \frac{D_3E_3F_3}{r_3} = \frac{\Delta}{2R} \qquad (5)$$

$$D_1E_1F_1 + D_2E_2F_2 + D_3E_3F_3 - DEF = 2\Delta \qquad (6)$$

$$DEF \cdot D_1E_1F_1 \cdot D_2E_2F_2 \cdot D_3E_3F_3 = \frac{\Delta^6}{16R^4} \qquad (7)$$

$$\frac{1}{D_1E_1F_1} + \frac{1}{D_2E_2F_2} + \frac{1}{D_3E_3F_3} - \frac{1}{DEF} = 0 \qquad (8)$$

$$\left.\begin{aligned} \frac{AEF \cdot BFD \cdot CDE}{IEF \cdot IFD \cdot IDE} &= \frac{\Delta^2}{r^4} \\ \frac{AE_1F_1 \cdot BF_1D_1 \cdot CD_1E_1}{I_1E_1F_1 \cdot I_1F_1D_1 \cdot I_1D_1E_1} &= \frac{\Delta^2}{r_1^4} \end{aligned}\right\} \qquad (9)$$

and so on.

$$\frac{AEF \cdot BFD \cdot CDE}{(DEF)^2} = \frac{AE_1F_1 \cdot BF_1D_1 \cdot CD_1E_1}{(D_1E_1F_1)^2} = \cdots\cdots = \frac{\Delta}{4} \qquad (10)$$

$$\Delta_0 = 2Rs, \quad \Delta_1 = 2Rs_1, \quad \Delta_2 = 2Rs_2, \quad \Delta_3 = 2Rs_3 \qquad (11)$$

$$\Delta_0\Delta_1\Delta_2\Delta_3 = 16R^4\Delta^2 \qquad (12)$$

$$\left.\begin{aligned} \Delta_0 : \Delta &= 2R : r \\ \Delta_1 : \Delta &= 2R : r \end{aligned}\right\} \qquad (13)$$

and so on.

$$\frac{\Delta}{\Delta_1}+\frac{\Delta}{\Delta_2}+\frac{\Delta}{\Delta_3}-\frac{\Delta}{\Delta_0}=2 \tag{14}$$

$$\left.\begin{aligned} 2\Delta\Delta_0 &= abcs = 4Rr_1r_2r_3 \\ 2\Delta\Delta_1 &= abcs_1 = 4Rr\,r_2r_3 \end{aligned}\right\} \tag{15}$$

and so on.

$$2r\Delta_0 = 2r_1\Delta_1 = 2r_2\Delta_2 = 2r_3\Delta_3 = abc \tag{16}$$

$$\left.\begin{aligned} \Delta_0 : \Delta &= \Delta : DEF \\ \Delta_1 : \Delta &= \Delta : D_1E_1F_1 \end{aligned}\right\} \tag{17}$$

and so on.

$$\left.\begin{aligned} \Delta_0 : DEF &= 4R^2 : r^2 \\ \Delta_1 : D_1E_1F_1 &= 4R^2 : r_1^2 \end{aligned}\right\} \tag{18}$$

and so on.

$$\left.\begin{aligned} \Delta_0^2 &= R\,(r_2+r_3)\,(r_3+r_1)\,(r_1+r_2) \\ \Delta_1^2 &= R\,(r_2+r_3)\,(r_3-r)\,(r_3-r) \\ \Delta_2^2 &= R\,(r_3+r_1)\,(r_3-r)\,(r_1-r) \\ \Delta_3^2 &= R\,(r_1+r_2)\,(r_1-r)\,(r_2-r) \end{aligned}\right\} \tag{19}$$

Corresponding expressions may be obtained for

ABC, HCB, CHA, BAH

by substituting instead of

$$R,\quad r,\quad r_1,\quad r_2,\quad r_3$$

$$\tfrac{1}{2}R,\quad \rho,\quad \rho_1,\quad \rho_2,\quad \rho_3$$

$$\left.\begin{aligned} 4\Delta_0 &= 2(\alpha_2+\alpha_3)\,\alpha_1 &&= 2(\beta_3+\beta_1)\,\beta_2 &&= 2(\gamma_1+\gamma_2)\,\gamma_3 \\ &= (\alpha_1-\alpha)(\alpha_2+\alpha_3) + &&(\beta_2-\beta)(\beta_3+\beta_1) + &&(\gamma_3-\gamma)(\gamma_1+\gamma_2) \\ 4\Delta_1 &= 2(\alpha_2+\alpha_3)\,\alpha &&= 2(\beta_2-\beta)\,\beta_3 &&= 2(\gamma_3-\gamma)\,\gamma_2 \\ &= -(\alpha_1-\alpha)(\alpha_2+\alpha_3) + &&(\beta_2-\beta)(\beta_3+\beta_1) + &&(\gamma_3-\gamma)(\gamma_1+\gamma_2) \\ 4\Delta_2 &= 2(\alpha_1-\alpha)\,\alpha_2 &&= 2(\beta_3+\beta_1)\,\beta &&= 2(\gamma_3-\gamma)\,\gamma_1 \\ &= (\alpha_1-\alpha)(\alpha_2+\alpha_3) - &&(\beta_2-\beta)(\beta_3+\beta_1) + &&(\gamma_3-\gamma)(\gamma_1+\gamma_2) \\ 4\Delta_3 &= 2(\alpha_1-\alpha)\,\alpha_2 &&= 2(\beta_2-\beta)\,\beta_1 &&= 2(\gamma_1+\gamma_2)\,\gamma \\ &= (\alpha_1-\alpha)(\alpha_2+\alpha_3) + &&(\beta_2-\beta)(\beta_3+\beta_1) - &&(\gamma_3-\gamma)(\gamma_1+\gamma_2) \end{aligned}\right\} \tag{20}$$

$$\left.\begin{aligned}
\mathrm{I_1BC} &= \frac{ar_1}{2} = \frac{r_1^2(r_2+r_3)}{2\sqrt{r_2r_3+r_3r_1+r_1r_2}} \\
\mathrm{AI_2C} &= \frac{br_2}{2} = \frac{r_2^2(r_3+r_1)}{2\sqrt{r_2r_3+r_3r_1+r_1r_2}} \\
\mathrm{ABI_3} &= \frac{cr_3}{2} = \frac{r_3^2(r_1+r_2)}{2\sqrt{r_2r_3+r_3r_1+r_1r_2}}
\end{aligned}\right\} \quad (21)$$

The three preceding triangles are similar to $I_1I_2I_3$ and expressions for them may be derived from the expressions for Δ_0 by the comparison of homologous lines in the triangles.

Thus in I_1BC, $I_1I_2I_3$ the perpendiculars I_1D_1, I_1A, or r_1, α_1, are homologous lines;

therefore $$I_1BC : \Delta_0 = r_1^2 : \alpha_1^2.$$

Similarly expressions may be found for

triangles	which are similar to
IBC, AI_3C, ABI_2	II_3I_2 or Δ_1
I_3BC, AIC, ABI_1	I_3II_1 or Δ_2
I_2BC, AI_1C, ABI	I_2I_1I or Δ_3.

From these again, by making the appropriate changes in the letters, corresponding expressions may be found for

triangles	which are similar to
AYZ, XBZ, XYC	ABC
HYZ, XCZ, XYB	HCB
CYZ, XHZ, XYA	CHA
BYZ, XAZ, XYH	BAH.

Triangles connected with the Angular Bisectors.

$$\left.\begin{aligned}
\mathrm{LMN} &= \frac{2abc\Delta}{(b+c)(c+a)(a+b)} = \frac{l_1 l_2 l_3}{4s}\\
\mathrm{LM'N'} &= \frac{2abc\Delta}{(b+c)(a-c)(a-b)} = \frac{l_1\lambda_2\lambda_3}{4s_1}\\
\mathrm{L'MN'} &= \frac{2abc\Delta}{(b-c)(c+a)(a-b)} = \frac{\lambda_1 l_2\lambda_3}{4s_2}\\
\mathrm{L'M'N} &= \frac{2abc\Delta}{(b-c)(a-c)(a+b)} = \frac{\lambda_1\lambda_2 l_3}{4s_3}
\end{aligned}\right\} \quad (22)$$

$$\left.\begin{aligned}
\mathrm{LL'M} &= \frac{2abc\Delta}{(c+a)(b^2-c^2)} \quad \mathrm{LL'N} = \frac{2abc\Delta}{(a+b)(b^2-c^2)}\\
\mathrm{MM'N} &= \frac{2abc\Delta}{(a+b)(a^2-c^2)} \quad \mathrm{MM'L} = \frac{2abc\Delta}{(b+c)(a^2-c^2)}\\
\mathrm{NN'L} &= \frac{2abc\Delta}{(b+c)(a^2-b^2)} \quad \mathrm{NN'M} = \frac{2abc\Delta}{(c+a)(a^2-b^2)}
\end{aligned}\right\} \quad (23)$$

$$\left.\begin{aligned}
\mathrm{LL'M'} &= \frac{2abc\Delta}{(a-c)(b^2-c^2)} \quad \mathrm{LL'N'} = \frac{2abc\Delta}{(a-b)(b^2-c^2)}\\
\mathrm{MM'N'} &= \frac{2abc\Delta}{(a-b)(a^2-c^2)} \quad \mathrm{MM'L'} = \frac{2abc\Delta}{(b-c)(a^2-c^2)}\\
\mathrm{NN'L'} &= \frac{2abc\Delta}{(b-c)(a^2-b^2)} \quad \mathrm{NN'M'} = \frac{2abc\Delta}{(a-c)(a^2-b^2)}
\end{aligned}\right\} \quad (24)$$

$$\left.\begin{aligned}
&\mathrm{LMN} : \Delta_0 = 4s_1s_2s_3 : (b+c)(c+a)(a+b)\\
&\mathrm{LM'N'} : \Delta_1 = 4s\,s_2s_3 : (b+c)(a-c)(a-b)\\
&\mathrm{L'MN'} : \Delta_2 = 4s\,s_3s_1 : (b-c)(c+a)(a-b)\\
&\mathrm{L'M'N} : \Delta_3 = 4s\,s_1s_2 : (b-c)(a-c)(a+b)
\end{aligned}\right\} \quad (25)$$

Triangles connected with the Orthocentre.

$$\left.\begin{aligned} ABX &= \frac{(a^2-b^2+c^2)\Delta}{2a^2} \quad & ACX &= \frac{(a^2+b^2-c^2)\Delta}{2a^2} \\ BCY &= \frac{(a^2+b^2-c^2)\Delta}{2b^2} \quad & BAY &= \frac{(-a^2+b^2+c^2)\Delta}{2b^2} \\ CAZ &= \frac{(-a^2+b^2+c^2)\Delta}{2c^2} \quad & CBZ &= \frac{(a^2-b^2+c^2)\Delta}{2c^2} \end{aligned}\right\} \tag{26}$$

$$\left.\begin{aligned} \Delta_a = HCB &= \frac{(a^2-b^2+c^2)(a^2+b^2-c^2)}{16\Delta} \\ \Delta_b = CHA &= \frac{(a^2+b^2-c^2)(-a^2+b^2+c^2)}{16\Delta} \\ \Delta_c = BAH &= \frac{(-a^2+b^2+c^2)(a^2-b^2+c^2)}{16\Delta} \end{aligned}\right\} \tag{27}$$

$$\left.\begin{aligned} HBX &= \frac{(a^2-b^2+c^2)^2(a^2+b^2-c^2)}{32a^2\Delta} \\ HCX &= \frac{(a^2-b^2+c^2)(a^2+b^2-c^2)^2}{32a^2\Delta} \end{aligned}\right\} \tag{28}$$

$$\left.\begin{aligned} AYZ = \frac{(-a^2+b^2+c^2)^2\Delta}{4b^2c^2}, \quad XBZ = \frac{(a^2-b^2+c^2)^2\Delta}{4c^2a^2}, \\ XYC = \frac{(a^2+b^2-c^2)^2\Delta}{4a^2b^2} \end{aligned}\right\} \tag{29}$$

$$XYZ = \frac{(-a^2+b^2+c^2)(a^2-b^2+c^2)(a^2+b^2-c^2)\Delta}{4a^2b^2c^2} \tag{30}$$

$$\left.\begin{aligned} DEF : XYZ &= r \ : 2\rho \\ D_1E_1F_1 : XYZ &= r_1 \ : 2\rho \end{aligned}\right\} \tag{31}$$

and so on.

$$\left.\begin{aligned}\Delta_0 : \Delta &= 4s : x+y+z \\ \Delta_1 : \Delta &= 4s_1 : x+y+z\end{aligned}\right\} \qquad (32)$$

and so on.

Miscellaneous.

$$\left.\begin{aligned}\mathrm{I\,I_1O} \cdot 8\Delta &= abc\,(c-b) \\ \mathrm{I\,I_2O} \cdot 8\Delta &= abc\,(c-a) \\ \mathrm{I\,I_3O} \cdot 8\Delta &= abc\,(a-b)\end{aligned}\right\} \qquad (33)$$

$$\left.\begin{aligned}\mathrm{I\,I_1H} \cdot 8\Delta &= a(c-b)\,(b^2+c^2-a^2) \\ \mathrm{I\,I_2H} \cdot 8\Delta &= b(c-a)\,(c^2+a^2-b^2) \\ \mathrm{I\,I_3H} \cdot 8\Delta &= c(a-b)\,(a^2+b^2-c^2)\end{aligned}\right\} \qquad (34)$$

$$\left.\begin{aligned}(2\,\mathrm{I_2I_3I} - \mathrm{I_2I_3O})\,8\Delta &= abc\,(b+c) \\ (2\,\mathrm{I_3I_1I} - \mathrm{I_3I_1O})\,8\Delta &= abc\,(c+a) \\ (2\,\mathrm{I_1I_2I} - \mathrm{I_1I_2O})\,8\Delta &= abc\,(a+b)\end{aligned}\right\} \qquad (35)$$

$$\left.\begin{aligned}(2\,\mathrm{I_2I_3I} - \mathrm{I_2I_3H})\,8\Delta &= a(b+c)\,(b^2+c^2-a^2) \\ (2\,\mathrm{I_3I_1I} - \mathrm{I_3I_1H})\,8\Delta &= b(c+a)\,(c^2+a^2-b^2) \\ (2\,\mathrm{I_1I_2I} - \mathrm{I_1I_2H})\,8\Delta &= c(a+b)\,(a^2+b^2-c^2)\end{aligned}\right\} \qquad (36)$$

$$\left.\begin{aligned}\mathrm{I\,H\,O} \cdot 8\Delta &= s\,(c-b)\,(c-a)\,(a-b) \\ \mathrm{I_1H\,O} \cdot 8\Delta &= s_1(c-b)\,(c+a)\,(a+b) \\ \mathrm{I_2H\,O} \cdot 8\Delta &= s_2(b+c)\,(c-a)\,(a+b) \\ \mathrm{I_3H\,O} \cdot 8\Delta &= s_3(b+c)\,(c+a)\,(a-b)\end{aligned}\right\} \qquad (37)$$

$$\left.\begin{array}{l} \mathrm{I\,I_1O} \cdot 8\Delta = (r_1 - r)(r_3 - r_2)(r_2r_3 + rr_1) \\ \text{and so on.} \end{array}\right\} \quad (38)$$

$$\left.\begin{array}{l} \mathrm{I\,I_1H} \cdot 4\Delta = (r_1 - r)(r_3 - r_2)(r_2r_3 - rr_1) \\ \text{and so on.} \end{array}\right\} \quad (39)$$

$$\left.\begin{array}{l} (2\,\mathrm{I_2I_3I} - \mathrm{I_2I_3O})\,8\Delta = (r + r_1)(r_2 + r_3)(r_2r_3 + rr_1) \\ \text{and so on.} \end{array}\right\} \quad (40)$$

$$\left.\begin{array}{l} (2\,\mathrm{I_2I_3I} - \mathrm{I_2I_3H})\,4\Delta = (r + r_1)(r_2 + r_3)(r_2r_3 - rr_1) \\ \text{and so on.} \end{array}\right\} \quad (41)$$

$$\left.\begin{array}{l} \mathrm{I\,H\,O} \cdot 32\Delta = \dfrac{1}{\mathrm{R}r}(r_1 - r_2)(r_1 - r_3)(r_2 - r_3)(r_1 - r)(r_2 - r)(r_3 - r) \\ \mathrm{I_1H\,O} \cdot 32\Delta = \dfrac{1}{\mathrm{R}r_1}(r_1 + r_2)(r_1 + r_3)(r_3 - r_2)(r_1 - r)(r_2 + r)(r_3 + r) \\ \text{and so on.} \end{array}\right\} (42)$$

It is probable that some of the preceding 42 formulae may belong to earlier dates, and to other authors than those indicated below. I shall be glad if any reader, who knows of earlier sources than those I have recorded, will take the trouble to inform me.

(2), (3) Mr E. Hain in *Nouvelle Correspondance Mathématique*, I. 75 (1874-5).

(5), (6) Feuerbach, *Eigenschaften...des...Dreiecks*, §§ 8, 9 (1822).

(7), (8) Mr B. Möllmann in Grunert's *Archiv*, XVII., 396 (1851).

(9), (10) Mr Combier gives the first of each of these expressions in the *Journal de Mathématiques Élémentaires*, III., 351 (1879).

(11), (12), (14) Mr B. Möllmann in Grunert's *Archiv*, XVII., 393-6 (1851).

(13) C. J. Matthes gives the first expression in *Commentatio de Proprietatibus Quinque Circulorum*, p. 55 (1831); T. S. Davies gives the others in the *Lady's and Gentleman's Diary* for 1842, p. 87.

(15) The first expression is given by Mr A. R. Clarke in 1847 in the *Mathematician*, III., 45 (1856); all are given by Mr C. Hellwig in Grunert's *Archiv*, XIX., 43 (1852).

(17) These proportions are implied in Feuerbach, *Eigenschaften...des...Dreiecks*, § 61 (1822). The first of them is given by C. J. Matthes in his *Commentatio*, p. 55, and also the first of (18).

(19) The first expression is given by C. Adams in his *Eigenschaften...des... Dreiecks*, p. 61 (1846); the corresponding expression for ABC is given on p. 62.

(20) The last values of each of these sets are given by Thomas Weddle in the *Lady's and Gentleman's Diary* for 1845, p. 75.

(21) T. S. Davies in the *Lady's and Gentleman's Diary* for 1842, p. 88.

(22)-(25) Mr Georges Dostor in the *Journal de Mathématiques Élémentaires et Spéciales*, IV. 21-23 (1880). The first expression for LMN, however, is given by Grunert in his article "Dreieck," quoted on p. 25. C. F. A. Jacobi, *De Triangulorum Rectilineorum Proprietatibus*, p. 15 (1825), gives the expression

$$\frac{2\triangle}{(a+b+c)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})-1}.$$

(26)-(28) Mr C. Hellwig in Grunert's *Archiv*, XIX., 25-26 (1852).

(30) Feuerbach, *Eigenschaften...des...Dreiecks*, §23 (1822).

(31)-(32) C. Adams, *Eigenschaften...des...Dreiecks*, pp. 54, 53 (1846).

(33)-(42) Mr C. Hellwig in Grunert's *Archiv*, XIX., 43-50 (1852).

Third Meeting, May 11th, 1883.

JOHN STURGEON MACKAY, Esq., M.A., President, in the Chair.

The Nine-point Circle.

By DAVID MUNN, F.R.S.E.

An Account of Newton's "Opticks."

By CARGILL G. KNOTT, D.Sc., F.R.S.E.

Fourth Meeting, June 8th, 1883.

JOHN STURGEON MACKAY, Esq., M.A., President, in the Chair.

New Proof of Professor Tait's Problem of Arrangement. Some short notes of interest to Teachers.

BY THOMAS MUIR, M.A.

The Fundamental Notions of the Differential Calculus.

BY A. Y. FRASER, M.A.

Plücker's first equation connecting the singularities of Curves.

By CARGILL G. KNOTT, D.Sc.

[The following notes are exactly as I left them in the hands of the Committee of the Society eleven years ago. They are printed now in the hope that, chiefly because of their brevity, they may be found useful to members, who may not have leisure or opportunity to read up the subject in the recognised text-books. —Jan., 1894. C. G. K.]

Let $U = f(x, y, z) = 0$ be the equation in trilinear co-ordinates of a curve of the n^{th} degree.

The number of terms is obviously

$$1 + 2 + 3 + \ldots\ldots + (n+1) = \tfrac{1}{2}(n+1)(n+2).$$

Hence $\frac{1}{2}(n+1)(n+2) - 1 = \frac{1}{2}n(n+3)$ points determine a general curve of the n^{th} degree.

According to Plücker, a curve may be considered as kinematically described by a point which moves along a line which continually rotates about that point.

This conception gives very simple notions as to the nature of singularities—double points, cusps, points of inflexion, double tangents, and so on.

Of these, only the cusp and point of inflexion are true singularities, the former being produced by a stationary point, the latter by a stationary line.

Still, a singular point may be more generally defined as a point which has descriptive properties not, in general, possessed by other points.

For example, a double point has two tangents, a triple point three, etc., a double tangent touches the curve in two distinct points, or, more strictly, meets the curve in two pairs of coincident points.

In the equation $\qquad U = f(xyz) = 0$

substitute for x the expression $\lambda x + \mu x_1$, for y, $\lambda y + \mu y_1$, and for z, $\lambda z + \mu z_1$. Then by Taylor's theorem U becomes

$$\lambda^n U + \lambda^{n-1}\mu\nabla U + \lambda^{n-2}\mu^2\frac{1}{\lfloor 2}\nabla^2 U + \ldots + \frac{\mu^n}{\lfloor n}\nabla^n U$$

where $$\nabla = x_1\frac{\partial}{\partial x} + y_1\frac{\partial}{\partial y} + z_1\frac{\partial}{\partial z}.$$

This gives n values of λ/μ if equated to zero. That is, any arbitrary line cuts the curve in n points real or imaginary.

If the point xyz is on the curve as well as the point ($\lambda x + \mu x'$, etc.) then $U = 0$, and there are $(n-1)$ other points of intersection.

If, further, $\nabla U = 0$, then there are two coincident points ($\lambda/\mu = 0$), and

$$\nabla U = x_1\frac{\partial U}{\partial x} + y_1\frac{\partial U}{\partial y} + z_1\frac{\partial U}{\partial z} = 0$$

is the equation of the tangent at the point (xyz), $x_1\ y_1\ z_1$ being any point on the tangent.

If the point $x_1y_1z_1$ is fixed, the equation $\nabla U = 0$ of degree $(n-1)$ in xyz represents the First Polar, a curve of the $(n-1)^{th}$ degree cutting the curve

$$U = 0$$

in $n(n-1)$ points, which include all the points to which tangents can be drawn from the points $x_1y_1z_1$.

The greatest number of tangents which can be drawn from any point to the curve, is called the *class* of the curve.

In the general curve of the n^{th} degree the class is $n(n-1)$; but if singularities exist the class is not so great.

But the equation $\nabla U = 0$ may be true whatever $x_1y_1z_1$ may be *i.e.*, if

$$\frac{\partial U}{\partial x} = 0, \quad \frac{\partial U}{\partial y} = 0, \quad \frac{\partial U}{\partial z} = 0 \text{ at } x, y, z.$$

That is, any line through xyz cuts the curve in two coincident points, or in a double point.

Hence $\nabla U = 0$ must pass through all the double points.

Or the First Polar passes through the double points of a curve.

Hence, if there are δ double points, since each double point counts for two, the other intersections cannot be greater than

$$n(n-1) - 2\delta,$$

and this is of course a superior limit to the number of tangents than can be drawn from a given point to the curve.

Consider more closely the conditions

$$\frac{\partial U}{\partial x} = 0, \quad \frac{\partial U}{\partial y} = 0, \quad \frac{\partial U}{\partial z} = 0.$$

Here there is, generally speaking, if $\nabla^2 U$ does not identically vanish, a node. The particular character of the node is got by considering the properties of $\nabla^2 U$.

If $\nabla^2 U = 0$ as well as $U = 0$, $\nabla U = 0$, there are three coincident points.

For a double point the equation $\nabla^2 U = 0$ gives two lines—two values of $x_1 y_1 z_1$.

In certain cases, however, these tangent lines coincide, namely, when

$$\nabla^2 U \text{ is a complete square,}$$

i.e., when

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 = Z = 0$$

$$\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial z^2} - \left(\frac{\partial^2 u}{\partial y \partial z}\right)^2 = X = 0$$

$$\frac{\partial^2 u}{\partial z^2}\frac{\partial^2 u}{\partial x^2} - \left(\frac{\partial^2 u}{\partial z \partial x}\right)^2 = Y = 0.$$

But in this case the tangent to any First Polar

$$\nabla U = 0$$

is given by the condition

$$\left(x'\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + z'\frac{\partial}{\partial z} \right) \nabla U = 0$$

independently of the values $x_1 y_1 z_1$ in ∇U

or

$$\begin{vmatrix} \frac{\partial^2 U}{\partial x^2} & \frac{\partial^2 U}{\partial x \partial y} & \frac{\partial^2 U}{\partial x \partial z} \\ \frac{\partial^2 U}{\partial x \partial y} & \frac{\partial^2 U}{\partial y^2} & \dots \\ \dots & \dots & \dots \end{vmatrix} = 0 = H$$

must be satisfied. But evidently $X = 0$, $Y = 0$, $Z = 0$ satisfy this.

Hence at a cusp every first polar passes through the curve and has the same tangent with it—that is, meets the curve in three coincident points.

Hence if there are κ cusps, each cusp counts for three points, and the other points of intersection, or, what is the same thing, the *class* of the curve, is

$$n(n-1) - 2\delta - 3\kappa.$$

The equation $H = 0$ is called the Hessian. Its order is $3(n-2)$. It intersects the curve in $3n(n-2)$ points; and if there are no nodes or cusps these are points of inflexion of the original curve and the Hessian meets the curve in three coincident points.

Hence, generally, if ι = number of inflexions

$$\iota = 3n(n-2).$$

But if there are nodes and cusps, this number ι is reduced. The analytical condition for the Hessian is

$$U = 0, \ \nabla U = 0, \ \nabla^2 U = 0.$$

But for a double point U and ∇U vanish independently of $x_1 y_1 z_1$; and the polar conic $\nabla^2 U = 0$ reduces to the two tangents at

the double point; hence with these conditions $x_1y_1z_1$ may be a point anywhere on either of the tangents. ∇U vanishes because

$$\frac{\partial U}{\partial x}=0 \quad \frac{\partial U}{\partial y}=0 \quad \frac{\partial U}{\partial z}=0$$

or

$$\left.\begin{array}{l} x\dfrac{\partial^2 U}{\partial x^2}+y\dfrac{\partial^2 U}{\partial x\partial y}+z\dfrac{\partial^2 U}{\partial x\partial z}=(n-1)\dfrac{\partial U}{\partial x}=0 \\ \quad \ldots \text{ etc. } \ldots \qquad \ldots \qquad \ldots \qquad =0 \\ \quad \ldots \text{ etc. } \ldots \qquad \ldots \qquad \ldots \qquad =0 \end{array}\right\}$$

Whence eliminating xyz

$$H=0, \text{ the Hessian.}$$

Or the Hessian also passes through all the double points.

But, the general condition of there being three coincident points in which H intersects U obviously requires that H must touch U there, must have the two tangents also two tangents, must itself have a double point there.

Hence a double point counts for 6 intersections between H and U.

For rigid proof, take the node as origin. Hence U contains no term in x and y lower than the second degree. Consider the lowest dimensions of $H=0$ in x and y, thus:

$$\begin{vmatrix} \dfrac{\partial^2 U}{\partial x^2} & \dfrac{\partial^2 U}{\partial x\partial y} & \dfrac{\partial^2 U}{\partial x\partial z} \\ \dfrac{\partial^2 U}{\partial x\partial y} & \dfrac{\partial^2 U}{\partial y^2} & \dfrac{\partial^2 U}{\partial y\partial z} \\ \dfrac{\partial^2 U}{\partial x\partial z} & \dfrac{\partial^2 U}{\partial y\partial z} & \dfrac{\partial^2 U}{\partial z^2} \end{vmatrix} \quad \text{is} \quad \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

Hence the order of the lowest terms in x and y is 2. Therefore $H=0$ has also a double point at the origin.

Again, let the tangent at the origin be the line x. Then x will be a factor in the equation $U=0$ and must be present in the expressions $\dfrac{\partial^2 U}{\partial y^2}$, $\dfrac{\partial^2 U}{\partial z^2}$, $\dfrac{\partial^2 U}{\partial y\partial z}$, one of which is present in every term of $H=0$. Hence x is also a tangent to H.

Hence if there are δ nodes

$$\iota = 3n(n-2) - 6\delta.$$

Take now the case of the cusp. Take it as origin and let $x=0$ be the tangent. Then x^2 is a factor in $U=0$. Hence the lowest dimensions in x and y of H are

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{vmatrix}$$

Hence 3 is the order of the lowest term in $H=0$, and each term contains x^2 as a factor. Hence this point is a triple point on H formed by a simple branch passing through a cusp; and the coincident tangents coincide with the cuspidal tangent. But a triple point and a double point meet in 6 coincident points, and the common pair of tangents means other two points. Hence H cuts U at a cusp in 8 points.

Hence if there are δ nodes and κ cusps

$$\iota = 3n(n-2) - 6\delta - 8\kappa.$$

The possible double points to the curve are limited in number; they cannot exceed $\frac{1}{2}(n-1)(n-2)$. For, if possible, let there be $\frac{1}{2}(n-1)(n-2)+1$.

Then, these points together with $(n-3)$ other points will determine a curve of degree $(n-2)$ because these together give

$$\tfrac{1}{2}(n-2)(n+1) = \tfrac{1}{2}\{n-2\}\{n-2+3\}.$$

Hence this curve of degree $(n-2)$ intersects the curve of degree n in

$$(n-1)(n-2)+2+n-3 \text{ points}$$
$$=(n-1)^2 = n(n-2)+1$$

which is impossible.

The number

$$D = \tfrac{1}{2}(n-1)(n-2) - \delta - \kappa$$

is called the "deficiency" of the curve. It is of great importance in the theory of transformations.

The curve for which $D = 0$ is unicursal.

The curve for which $D = 1$ is bicursal, and so on.

Fifth Meeting, July 13th, 1883.

JOHN STURGEON MACKAY, Esq., M.A., President, in the Chair.

Some notes on Quaternions.

By CARGILL G. KNOTT, D.Sc., F.R.S.E.

Some theorems on Radical Axes.

By DAVID MUNN, F.R.S.E.

Edinburgh Mathematical Society.

LIST OF MEMBERS.

FIRST SESSION, 1883.

1 R. E. ALLARDICE, M.A., Assistant to the Professor of Mathematics, University, Edinburgh.

A. J. G. BARCLAY, M.A., Mathematical Master, George Watson's College, Edinburgh.

C. W. C. BARLOW, M.A., St Peter's College, Cambridge.

JAMES BLYTH, M.A., F.R.S.E., Professor of Mathematics and Natural Philosophy, Andersonian College, Glasgow.

5 JAMES BOLAM, Navigation School, Leith.

H. H. BROWNING, M.A., 14 Newton Place, Charing Cross, Glasgow.

J. R. BURGESS, M.A., Mathematical Master, Merchiston Castle School, Edinburgh.

Peter Campbell, Mathematical Master, George Watson's College, Edinburgh.

George Chrystal, M.A., F.R.S.E., Professor of Mathematics, University, Edinburgh, (*Hon. Member*).

10 W. Cooper Dickson, M.A., U.P. Divinity Hall, Edinburgh.

Archibald C. Elliot, B.Sc., University of Edinburgh.

James Fairweather, C.E., 12 Buccleuch Place, Edinburgh.

R. M. Ferguson, Ph.D., F.R.S.E., 8 Queen Street, Edinburgh.

A. Y. Fraser, M.A., Mathematical Master, George Watson's College, Edinburgh.

15 E. P. Frederick, M.A., Mathematical Master, Loretto.

R. P. Hardie, 4 Scotland Street, Edinburgh.

Wm. Harvey, M.A., 32 George Square, Edinburgh.

E. B. Kirk, (Rev.), Barrhead.

Cargill G. Knott, D.Sc., F.R S.E., Assistant to Professor of Natural Philosophy, Edinburgh University, (*Hon. Secretary and Treasurer*).

20 A. P. Laurie, M.A., King's College, Cambridge.

J. Barrie Low, M.A., Mathematical Master, Kelvinside Academy, Glasgow.

J. Macdonald, M.A., 11 South Charlotte Street, Edinburgh.

W. J. Macdonald, M.A., Mathematical Master, Daniel Stewart's College, Edinburgh.

A. Macfarlane, D.Sc., F.R.S.E., Examiner in Mathematics, University, Edinburgh, (*Vice-President*).

25 J. M'Kean, Lecturer on Mathematics, School of Arts, Edinburgh.

John S. Mackay, M.A., F.R.S.E., Mathematical Master, the Academy, Edinburgh, (*President*).

John Macmillan, M.A., B.Sc., F.R.S.E., Mathematical Master, Academy, Perth.

Charles M. Middleton, M.A., Mathematical Master, George Watson's College, Edinburgh.

Robert Miller, (Rev.), Mathematical Master, Ladies' College, George Square, Edinburgh.

30 William Milne, M.A., Mathematical Master, High School, Glasgow.

Robert Moodie, M.A., High School, Arbroath.

J. T. Morrison, 12 Meadow Place, Edinburgh.

Thomas Muir, M.A., F.R.S.E., Mathematical Master, High School, Glasgow.

David Munn, F.R.S.E., Mathematical Master, High School, Edinburgh.

35 Charles Niven, M.A., D.Sc., F.R.S., Professor of Natural Philosophy, University, Aberdeen, (*Hon. Member*).

R. T. Omond, The Observatory, Ben Nevis.

William Peddie, Assistant to the Professor of Natural Philosophy, University, Edinburgh.

Herbert B. Pickard, 3 Summerhall Square, Edinburgh.

Charles Prentice, C.A., 40 Castle Street, Edinburgh.

40 Wm. Raitt, M.A., B.Sc., School of Science and Art, Glasgow.

Thomas T. Rankin, C.E., Rector, Gartsherrie Science School, Coatbridge.

D. Rintoul, M.A., Corpus Christi College, Cambridge.

Wm. Ritchie, George Watson's College, Edinburgh.

Alex. Robertson, M.A., 30a St Andrew Square, Edinburgh.

45 Robert Robertson, M.A., Mathematical Master, Ladies' College, Queen Street, Edinburgh.

H. C. Robson, M.A., Sidney Sussex College, Cambridge.

J. R. Roxburgh, 23 North Mansion House Road, Edinburgh.

P. Guthrie Tait, M.A., Sec.R.S.E., Professor of Natural Philosophy, University, Edinburgh, (*Hon. Member*).

James Taylor, M.A., Dollar Institution.

50 George Thom, M.A., Rector, Dollar Institution.

Sir William Thomson, D.C.L., F.R.S., &c., Professor of Natural Philosophy, University, Glasgow, (*Hon. Member*).

William Thomson, M.A., Student in Divinity, Edinburgh.

David Traill, M.A., B.Sc., Mathematical Master, Academy, Edinburgh.

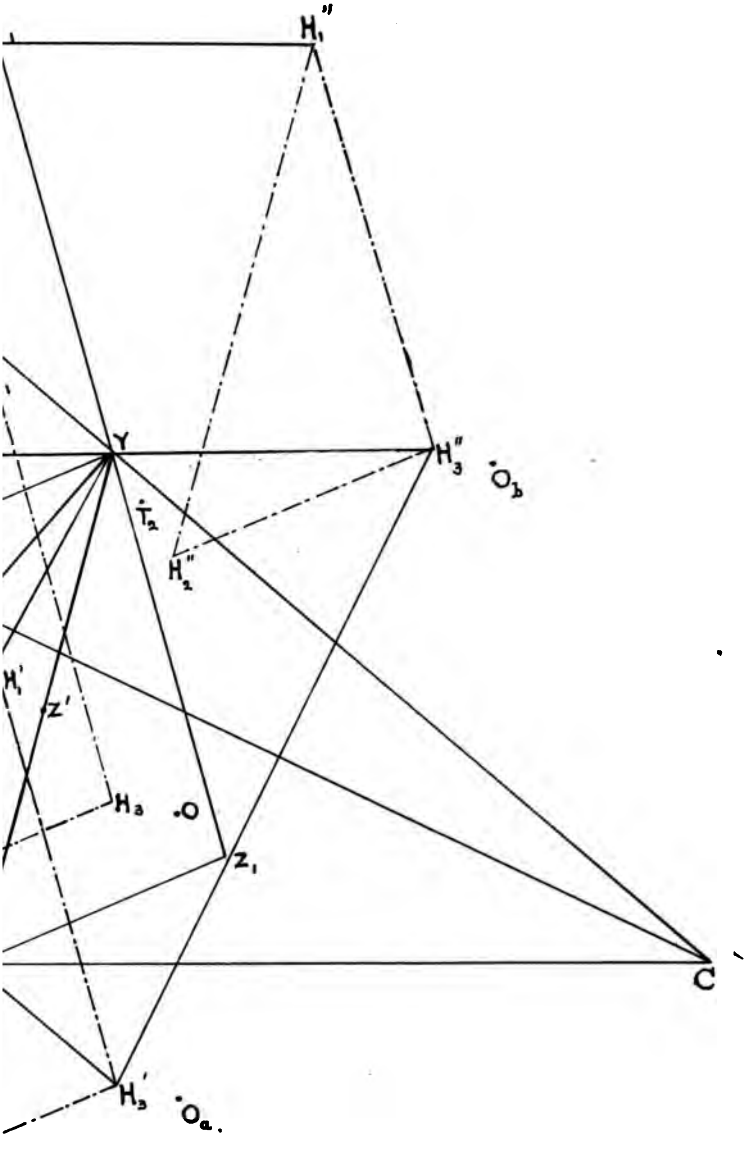
John E. Vernon, 11 Duncan Street, Edinburgh.

55 William Welsh, M.A., Fellow and Tutor, Jesus College, Cambridge.

Thomas White, 7 Claremont Place.

Laurence H. Wight, M.A., Lauder Road, Edinburgh.

58 Philip Wood, M.A., Head Master, Grammar School, Darlington.



H₁''
Y
H₃''
O_b
T₂
H₂''
H₁'
Z'
H₃
O
Z₁
C
H₃'
O_a.

PROCEEDINGS

OF THE

EDINBURGH MATHEMATICAL SOCIETY.

VOLUME II.

SESSION 1883-84.

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INDEX.

CORRIGENDA.

Page 4, line 5 from bottom—for skew, read crossed.
Page 12, § 13—delete the whole of this section.
Page 35, line 25—for I_4, read i_4.
" " —for *ux*, read *uv*.
Page 50, line 14—for fig. 32, read fig. 33.

PROCEEDINGS

OF THE

EDINBURGH MATHEMATICAL SOCIETY.

SECOND SESSION, 1883-84.

First Meeting, November 9th, 1883.

JOHN STURGEON MACKAY, Esq., M.A., F.R.S.E., President,
in the Chair.

Professor TAIT, University of Edinburgh, gave an introductory address on "Listing's Topologie."

This address will be found in *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* for January 1884, Fifth Series, vol. xvii., pp. 30–46.

For this Session the following Office-Bearers were elected:—

President—Mr THOMAS MUIR, M.A., F.R.S.E.

Vice-President—Mr A. J. G. BARCLAY, M.A.

Secretary and Treasurer—Mr A. Y. FRASER, M.A.

Committee—

Messrs R. E. ALLARDICE, M.A.; WM. PEDDIE, M.A.; ROBERT ROBERTSON, M.A.; DAVID TRAILL, M.A., B.Sc.

Second Meeting, December 14th, 1883.

THOMAS MUIR, Esq., M.A., F.R.S.E., President, in the Chair.

The Circles associated with the Triangle, viewed from their Centres of Similitude.

By JOHN STURGEON MACKAY, M.A., F.R.S.E.

[*Abstract.*]

The notation adopted in this paper for the triangle ABC is:

G the centroid.

I the centre of the inscribed circle.

I_1, I_2, I_3 the centres of the escribed circles within angles A, B, C.

O the centre of the circumscribed circle.

D, E, F; D_1, E_1, F_1; D_2, E_2, F_2; D_3, E_3, F_3 the points of contact of the inscribed and escribed circles with BC, CA, AB. The Ds lie all on BC, the Es on CA, and the Fs on AB.

H, K, L the mid points of BC, CA, AB.

X, Y, Z the feet of the perpendiculars from A, B, C, on BC, CA, AB.

A′, B′, C′ the vertices, opposite to A, B, C, of the triangle formed by drawing through A, B, C parallels to BC, CA, AB.

1. *a.* AX, BY, CZ are concurrent at O′.
   b. AD_1, BE_2, CF_3 „ „ I′.
   c. AD, BE_3, CF_2 „ „ I_1'.
   AD_3, BE, CF_1 „ „ I_2'.
   AD_2, BE_1, CF „ „ I_3'.

2. *a.* O′, orthocentre of $\triangle$ABC, is circumscribed centre of $\triangle$A′B′C′.
   b. I′, … centre* of $\triangle$ABC, is inscribed centre of $\triangle$A′B′C′.
   c. I_1', first … centre* of $\triangle$ABC, is first escribed centre of $\triangle$A′B′C′. Similarly for I_2' and I_3'.

3. *a.* Circumscribed centre of $\triangle$ABC is orthocentre of $\triangle$HKL.
   b Inscribed centre of $\triangle$ABC is … centre of $\triangle$HKL.
   c. First escribed centre of $\triangle$ABC is first … centre of $\triangle$HKL.

4. *a.* Centre of similitude of $\triangle$s ABC, HKL is found by joining AH and O′O, which intersect at G the centroid.

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* Two words are wanted to denote I′ and I_1' with respect to $\triangle$ABC.

b. Centre of similitude of $\triangle$s ABC, HKL is found by joining AH and I′I, which intersect at G the centroid.

c. Centre of similitude of $\triangle$s ABC, HKL is found by joining AH and $I_1'I_1$, which intersect at G the centroid.

5. *a*. AO′ is parallel to HO and = 2HO; O′G = 2OG.
b. AI′ „ „ HI and = 2HI; I′G = 2IG.
c. AI_1' „ „ HI_1 and = $2HI_1$; $I_1'G = 2I_1G$.

6. *a*. To find the circumscribed centre of $\triangle$HKL.
From GO′ cut off GM = $\frac{1}{2}$ GO. M is the point required.
b. To find the inscribed centre of $\triangle$HKL.
From GI′ cut off GJ = $\frac{1}{2}$ GI. J is the point required.
c. To find the first escribed centre of $\triangle$HKL.
From GI_1' cut off $GJ_1 = \frac{1}{2} GI_1$. J_1 is the point required.

7. *a*. M is the mid point of O′O, and O′ the external centre of similitude of the circumscribed circles of $\triangle$s ABC, HKL.
b. J is the mid point of I′I, and I′ the external centre of similitude of the inscribed circles of $\triangle$s ABC, HKL.
c. J_1 is the mid point of $I_1'I_1$, and I_1' the external centre of similitude of the first escribed circles of $\triangle$s ABC, HKL.

8. *a*. If on O′O there be taken to the right and left of G segments successively = half of those on the left and right, the points so determined will be circumscribed centres of successive median triangles. Process reversible.
b. If on I′I there be taken to the right and left of G segments successively = half of those on the left and right, the points so determined will be inscribed centres of successive median triangles. Process reversible.
c. If on $I_1'I_1$ there be taken to the right and left of G segments successively = half of those on the left and right, the points so determined will be first escribed centres of successive median triangles. Process reversible.

9. *a*. HM, KM, LM produced bisect O′A, O′B, O′C at U, V, W.
b. HJ, KJ, LJ „ „ I′A, I′B, I′C „ U, V, W.*
c. HJ_1, KJ_1, LJ_1 „ „ $I_1'A$, $I_1'B$, $I_1'C$ „ U, V, W.

10. *a*. Circumscribed circle of $\triangle$HKL is circumscribed circle of $\triangle$UVW.
b. Inscribed circle of $\triangle$HKL is inscribed circle of $\triangle$UVW.
c. First escribed circle of $\triangle$HKL is first escribed circle of $\triangle$UVW.

*** These three triads of points are all different, though denoted by the same letters.**

11. *a.* Six parallelograms, whose diagonals intersect at M are HOUO′, KOVO′, LOWO′; HKUV, KLVW, LHWU.
    b. Six parallelograms whose diagonals intersect at J are HIUI′, KIVI′, LIWI′; HKUV, KLVW, LHWU.
    c. Six parallelograms whose diagonals intersect at J_1 are HI_1UI_1', KI_1VI_1', LI_1WI_1'; HKUV, KLVW, LHWU.
12. *a.* HWKULV is a hexagon whose opposite sides are parallel, and respectively $= \frac{1}{2}O'A$, $\frac{1}{2}O'B$, $\frac{1}{2}O'C$.
    b. HWKULV is a hexagon whose opposite sides are parallel, and respectively $= \frac{1}{2}I'A$, $\frac{1}{2}I'B$, $\frac{1}{2}I'C$.
    c. HWKULV is a hexagon whose opposite sides are parallel, and respectively $= \frac{1}{2}I_1'A$, $\frac{1}{2}I_1'B$, $\frac{1}{2}I_1'C$.
13. *a.* AO′, BO′, CO′ pass through the points where the circumscribed circle of △HKL cuts the sides of △ABC.
    b. AI′, BI′, CI′ pass through the points where the inscribed circle of △HKL touches the sides of △HKL.
    c. AI_1', BI_1', CI_1' pass through the points where the first escribed circle of △HKL touches the sides of △HKL.

On Determinants with *p*-termed elements.

By Thomas Muir, M.A., F.R.S.E.

This paper will be found in the *Messenger of Mathematics* for January 1884, Vol. xiii, New Series.

Construction for Euclid II. 9, 10.

By R. W. M'Arthur.

Take line AB divided in C and D as in Euclid. On AD describe the rectangle AEFD having AE, DF each equal to AC or CB. According as D is in AB, or in AB produced, from DF or DF produced cut off FG equal to DB; and join CG, GE, EC.

Mr James Taylor gave a proof of the known theorem :—"If two sides of a skew quadrilateral ABDC inscribed in a circle be produced to meet in E, and FEG be drawn perpendicular to the diameter passing through E, the two other sides produced make equal intercepts on FEG." Mr Taylor's object was to call attention to the desirability of obtaining a simpler mode of demonstration.

Third Meeting, January 11*th*, 1884.

THOMAS MUIR, Esq., M.A., F.R.S.E., President, in the Chair.

Mathematical Models, chiefly of Surfaces of the Second Degree.

By Professor CHRYSTAL, University of Edinburgh.

Professor Chrystal exhibited a number of models made of wood, cardboard, thread, and plaster of Paris, and made use of them for the exposition of some of the principal properties of the surfaces represented.

Theorem relating to the Sum of selected Binomial-Theorem Co-efficients.

By Professor TAIT, University of Edinburgh.

This theorem will be found in the *Messenger of Mathematics* for February 1884, vol. xiii., New Series, p. 154.

Professor CHRYSTAL brought before the meeting a problem to which his attention had been drawn by Mr James Edward, M.A., B.Sc. The following is the problem, and Mr Edward's solution :—

Between two sides of a triangle to inflect a straight line which shall be equal to each of the segments of the sides between it and the base.

From AB, one of the sides of the given triangle ABC, cut off AD = AC, and join CD. Divide BC, internally at G and externally at K, in the ratio of AD to DC; on GK as diameter describe a circle cutting CD in P. Join BP; draw PF parallel to BA and meeting AC in F; and draw FE parallel to BP and meeting AB in E.

The circle GPK is the locus of the vertices of all the triangles on the base BC, and having their sides in the ratio BG : GC;

$\therefore$ BP : PC = BG : GC,
= AD : DC, (Construction)
= FP : PC; (Eucl. VI. 4)

$\therefore$ BP = FP, and BEFP is a rhombus.

But FC = FP, since AC = AD;

$\therefore$ BE = EF = FC.

Cor. 1. When triangle ABC is isosceles, EF is parallel to BC.

Cor. 2. When P moves up to D, F moves up to A. In this case, which is the limiting one for the point P within the triangle, BD = DA = AC. The limiting case therefore occurs when one of the sides is double of the other.

Cor. 3. When AB is greater than twice AC, the point P is outside the triangle, F is on CA produced, and, as before, BE = EF = FC.

Fourth Meeting, February 8th, 1884.

A. J. G. Barclay, Esq., M.A., Vice-President, in the Chair.

The Promotion of Research—A Presidential Address.

By Thomas Muir, M.A., F.R.S.E.

This paper has been printed by Mr Muir for distribution among the Members of the Society.

Illustrations of Harmonic Section.

By Hugh Hamilton Browning, M.A.

[*Abstract.*]

The object of the paper was to draw attention to a few important and well known cases of the harmonic section of a straight line, and to show their application to one or two problems of interest, more especially the method of drawing tangents to a conic by the ruler only. The effort throughout was to secure clearness, brevity, and freshness of proof, coupled with purely geometrical treatment.

Among other propositions were the following:

(*a*) O, P, V, W, X, are points in a straight line such that PV : PX $= OV^2 : OX^2$, and OP = PW; show that OV, OW, OX are in

harmonic progression, and apply the proof to show that any secant from a point outside a parabola is cut harmonically by the chord of contact of tangents from the point.

(*b*) O, P, V, W, C, X, P_1, are points in a straight line such that $CP^2 = CW \cdot CO$, $CP = CP_1$, and $OV^2 : OX^2 = CP^2 - CV^2 : CP^2 - CX^2$; show that OV, OW, OX are in harmonic progression, and apply the proof to the ellipse.

(*c*) A proof, by the reduction of the proportion when CP is less than CV and CX, suited to the hyperbola.

The importance of the fact that in an harmonic pencil any ray is the locus of the middle points of straight lines intercepted by the rays on either side of it and parallel to the fourth ray was illustrated by showing that as it holds when the rays are produced backwards, it immediately leads to such theorems as:

1. The intersection of the diagonals of a quadrilateral inscribed in a circle is upon the polar of the intersection of the opposite sides.
2. The theorem proposed by Mr James Taylor for simple proof at the Society's meeting on 14th December 1883.
3. The intersections of the tangents at the extremities of all chords of a conic which pass through one point lie on a straight line.

Among other proofs offered was the following, which shows that, with no further aid than that afforded by Euclid I.–III., tangents can be drawn to a circle with the ruler only.

Let BCED be a quadrilateral inscribed in a circle whose centre is K; let CB and ED meet at A, and CD, EB at O. Join OK, AK, and from O draw OR perpendicular to AK. Through the points B, C, O describe a circle, meeting AO again at X, and join CX.

Then $\angle CXO = \angle ABO = \angle ADC$;

therefore the points A, D, X, C are concyclic.

$$\begin{aligned} \text{Hence } AB \cdot AC = AO \cdot AX &= AO^2 + AO \cdot OX = AO^2 + CO \cdot OD, \\ &= AK^2 + OK^2 - 2AK \cdot KR + CO \cdot OD, \\ &= r^2 + AB \cdot AC + r^2 - CO \cdot OD - 2AK \cdot KR + CO \cdot OD; \end{aligned}$$

therefore $r^2 = AK \cdot KR$, a result which proves that O lies on the chord of contact of the tangents from A.

In a similar way another point may be found situated on the chord of contact; and thus the chord of contact is determined.

Note on a Theorem connected with the area of a $2n$-sided polygon.

By THOMAS MUIR, M.A., F.R.S.E.

The theorem is:—*If* $a_1, a_2, a_3, \ldots, a_{2n}$ *be the middle points of the sides of any convex polygon* $A_1A_2A_3 \ldots A_{2n}$ *then as regards areas*

$$a_1 a_2 \ldots a_{2n} = \tfrac{1}{2}A_1A_2 \ldots A_{2n} + \tfrac{1}{4}A_1A_3 \ldots A_{2n-1} + \tfrac{1}{4}A_2A_4 \ldots A_{2n}.$$

The following proof depends only on the theorem that the line joining the points of bisection of two sides of a triangle cuts off a triangle equal in area to a quarter of the original triangle. For convenience in writing, let us take the case where $n = 4$. Then

$$\left.\begin{array}{l} \tfrac{1}{4}A_1A_3A_5A_7 = \\ \qquad \tfrac{1}{4}(A_1A_2A_3 \ldots A_8 - A_1A_2A_3 - A_3A_4A_5 - A_5A_6A_7 - A_7A_8A_1) \\ \text{and } \tfrac{1}{4}A_2A_4A_6A_8 = \\ \qquad \tfrac{1}{4}(A_1A_2A_3 \ldots A_8 - A_2A_3A_4 - A_4A_5A_6 - A_6A_7A_8 - A_8A_1A_2) \end{array}\right\}$$

$$\left.\begin{array}{l} \therefore \tfrac{1}{4}A_1A_3A_5A_7 \\ + \tfrac{1}{4}A_2A_4A_6A_8 \end{array}\right\} = \tfrac{1}{2}A_1A_2A_3 \ldots A_8 + a_1 a_2 a_3 \ldots a_8 - A_1A_2A_3 \ldots A_8$$

and $\therefore a_1 a_2 a_3 \ldots a_8 = \tfrac{1}{2}A_1A_2A_3 \ldots A_8 + \tfrac{1}{4}A_1A_3A_5A_7 + \tfrac{1}{4}A_2A_4A_6A_8$ as was to be proved.

Fifth Meeting, March 14th, 1884.

A. J. G. BARCLAY, Esq., M.A., Vice-President, in the Chair.

Spherical Geometry.

By R. E. ALLARDICE, M.A.

The object of this paper is to bring together the principal properties of figures described on the surface of the sphere that can be established without the use of Solid Geometry or of Trigonometry.

The following properties of the spherical surface, which correspond to the definitions and axioms in Plane Geometry, are assumed. They may be considered as derived from one's general notion of the sphere.

a. On the surface of the sphere certain circles (great circles) can be drawn, which are closely analogous to straight lines in a plane. These great circles are all equal in circumference.

b. Through any two points one great circle can be drawn, and in general only one; if the distance between the two points be half a great circle, any number may be drawn.

c. Any two great circles intersect in two points (called antipodal points), the distance between which is half a great circle.

d. With any centre and any radius a circle may be described, called a small circle, unless the radius be a quadrant of a great circle, in which case the circle becomes a great circle.

e. Every circle, great or small, has two centres (or poles), these centres being antipodal points.

f. If two antipodal points move continuously on the sphere, they trace out what are called symmetric figures. These figures have corresponding elements equal, and are equal in area, but are not in general superposable. The one is, in fact, the perverse of the other.

An angle may be conceived as generated by the revolution of a great circular arc about a fixed point. Since the two characteristic properties of angles, which are that two equal angles are superposable and therefore identical, and that if a straight line trace out the whole (finite) angular space at a point it will return to its original position, are possessed also by circular arcs (a tracing point taking the place of a tracing line), arcs may evidently be treated as if they were angles, and arcs and angles may be spoken of as equal. The angle to which any arc corresponds is evidently the angle between the radii drawn to its extremities. From this it follows that the angle between two lines (great circular arcs) is equal to the angle between their middle points.

§ 1. The angle between two lines is equal or supplementary to the angle between their poles.

§ 2. The polar triangle. The triangle formed by joining the poles A', B', C' of the sides BC, CA, AB of the triangle ABC, (A' being the pole which lies on the same side of BC as A, and so on), is called the polar triangle of the triangle ABC. By § 1, $B'C'$, $C'A'$, $A'B'$ are either equal or supplementary to A, B, and C; and since motion from A' to B' corresponds to rotation from BC to the production of AC, the sides of the polar triangle must be the *supplements* of the angles of the original triangle. The polar property of the triangle is evidently reciprocal.

If both poles of each side of the original triangle be considered, eight triangles can be formed, the angles and sides of four of which are—(1) $\pi - a$, $\pi - b$, $\pi - c$, $\pi - A$, $\pi - B$, $\pi - C$; (2) $\pi - a$, b, c, $\pi - A$, B, C; (3) a, $\pi - b$, c, A, $\pi - B$, C; (4) a, b, $\pi - c$, A, B, $\pi - C$. The other four are the symmetric triangles. The triangles (2), (3), and (4) are the polars of the associated triangles of the original triangle, that is, the triangles formed by producing each pair of sides.

§ 3. The Principle of Polar Transformation.

From any theorem that has been established another theorem may be deduced by consideration of the polar figure. Thus the polar figure corresponding to a line passing through a point is a point lying on a line; and hence, if it has been proved that three lines l, m, n pass through the same point P, the three points L, M, N, the poles of l, m, n, all lie on the same line p, of which P is the pole. It must be noticed that to the internal bisector of an angle corresponds the external bisector of the corresponding line, that is, the point which bisects the supplement of the line, and which is a quadrant distant from the internal point of bisection. This follows from the fact that it is the supplements of the sides of the polar triangle that are equal to the angles of the original triangle.

§ 4. The area of a spherical triangle $= \dfrac{A+B+C-\pi}{\pi} \cdot \tfrac{1}{4}$ surface of sphere.

§ 5. $A+B+C > \pi < 3\pi$; $b+c > a$, &c.; $a+b+c < 2\pi$.

Since $A+B+C-\pi$ varies as area of triangle,

$\therefore A+B+C > \pi$.

In the polar triangle $a'+b'+c' > 0$;

$\therefore \pi - A + \pi - B + \pi - C > 0$; $\therefore A+B+C < 3\pi$.

Transforming the inequality $A'+B'+C' > \pi$ by means of the second polar triangle, there results

$\pi - a + b + c > \pi$; $\therefore b+c > a$.

Again in the polar triangle

$A'+B'+C' > \pi$; $\therefore \pi - a + \pi - b + \pi - c > \pi$; $\therefore a+b+c < 2\pi$.

§ 6. Theorems analogous to Euclid I. 4, 5, 6, 8, 15, 24, 25, and 26 (the first case only) can be proved for the sphere in much the same way as they are proved in Plane Geometry; but where there are congruent triangles in Plane Geometry, there may be either congruent or symmetric triangles in Spherical Geometry. Of these theorems No. 6 is the polar of No. 5, and the first case of No. 26 the polar of No. 4. Theorem 16 is only true with limitations, which make it almost useless. The second case of No. 26 is an ambiguous proposition in the case of the sphere, being the polar theorem of the ordinary "ambiguous case" of Plane Geometry.

§ 7. The polar theorem of Euclid I. 8. If two triangles have the three angles of the one equal to the three angles of the other, the triangles are either congruent or symmetric.

The polar theorems of Euclid I. 24 and I. 25.

§ 8. From the proposition that any two sides of a triangle are together greater than the third, which is proved above, there is easily deduced Euclid I. 19, the polar of which gives Euclid I. 18.

Note.—Since through two given points there can be drawn only one small circle of given radius and concave in a given direction, an arc of such a circle may be substituted in some of the above propositions for a great circular arc.

§ 9. Euclid, III. 7 and 8, true both for great circles and for small circles, and proved for both in the same way.

§ 10. All the theorems of the Third Book of Euclid are true for the sphere, with the following exceptions:—

(1.) That angles in the same segment of a circle are equal.

There is, however, a theorem analogous to this, which will be enunciated afterwards.

(2.) That the opposite angles of a quadrilateral inscribed in a circle are together equal to two right angles.

In the case of the sphere however, one pair of opposite angles of such a quadrilateral are together equal to the other pair

(3.) That the angle in a semicircle is a right angle, &c.

In the case of the sphere the triangle inscribed in a semicircle has its vertical angle equal to the sum of the other two, the triangle inscribed in a segment greater than a semicircle has its vertical angle less than the sum of the other two, and the triangle inscribed in a segment less than a semicircle its vertical angle greater than the sum of tne other two.

[The spherical triangle which has one angle equal to the sum of the other two, has many properties analogous to those of the right-angled plane triangle].

(4.) That the rectangle under the segments of secants passing through a fixed point is constant.

[In the sphere the product of the tangents of half the segments is constant].

Those of the above propositions that refer to angles reduce to the corresponding propositions of Plane Geometry if the condition be added that the three angles of a triangle are together equal to two right angles.

Definition. A spherical parallelogram is a quadrilateral which has its opposite sides equal.

§ 11. The opposite angles of a parallelogram are equal; the alternate angles made by the diagonals with the sides are equal; and the diagonals bisect one another.

§ 12. On a given base only one parallelogram can be described, having the side opposite this base in a given line. (Fig. 1.)

For if ABCD be a parallelogram, and AB and DC be produced to meet at E and F, then the triangles EAD and FCB are equiangular; $\therefore$ EA = CF, and ED = BF.

§ 13. Parallelograms on equal bases, and having a pair of opposite sides in the same lines, are equal. (Fig. 1.)

For $DD' = BB'$, $\angle D'DO = \angle B'BO'$, $\angle DD'O = \angle BB'O'$;
$\therefore \triangle DOD' = \triangle BO'B'$. Similarly, $OA'A = O'C'C$; $\therefore ABCD = A'B'C'D'$.

§ 14. If two parallel small circles be cut by a great circle in the points A, B, and C, D, then AC and BD are bisected by the great circle parallel to the two small circles, and the parts BA and CD intercepted by the small circles are equal. (Fig. 2.)

Draw OPO' a great circle perpendicular to ABDC; and draw the great circle OQO'.

Then $OQ = O'Q$, $OA = O'C$, $\angle OQA = \angle O'QC$; $\therefore AQ = QC$.
Again, $OF = O'G$; $\therefore BA = CD$.

Cor.—If the great circle ABCD touch one of the small circles, it must touch the other.

§ 15. The quadrilateral formed by joining the extremities of two equal arcs of equal and parallel small circles is a parallelogram. (Fig. 3.)

Let AB and CD be the arcs.

Draw the great circles ODO', OCO'.

Then $\angle CO'D = \angle COD = \angle AOB$.
$\therefore \angle AOD = \angle BOC$; and $AO = BO$, $OD = OC$.
$\therefore AD = BC$, and chord AB = chord CD; $\therefore$ ABCD is a parallelogram.

§ 16. Parallelograms on the same (or equal) bases, and between the same equal and parallel small circles, are equal in area. (Fig. 2.)

Let ABCD, $A'B'CD$ be the two parallelograms.

Then triangles $A'DA$, $B'CB$ are equal, &c. (as in Euclid I. 35).

Cor. 1.—Since the diagonal bisects a parallelogram, triangles on the same base and between the same parallel small circles are equal in area.

Cor. 2. From this it follows at once, that the locus of the vertex of a triangle of constant area on a fixed base is a small circle. (Lexell's Theorem.)

In order to find this small circle when one of the triangles ABC (fig. 4) is given, through B and C and through A two parallel and equal small circles must be drawn. Let O be the centre of the circle circumscribed to A′BC, O′ the point antipodal to O; then a circle with O′ as centre and O′A as radius is equal and parallel to the circle circumscribed to A′BC, and is the required locus.

Lexell's Theorem may also be proved as follows:—

If ABC be a triangle on a fixed base BC, and inscribed in a fixed small circle, then B + C — A is constant. (Fig. 5.)

[This is the analogue to Euclid III. 21, to which reference was made in § 10.]

Let BAC, BA′C be two of the triangles.

Then ABC + ACB — BAC = A′BC + A′CB — BA′C;

if BAC — ABA′ = BA′C — ACA′;

if CAO + A′BO = BA′O + ACO; which is true.

Now, let BAC (fig. 4) be one of the triangles of given area of Lexell's Theorem. Circumscribe a circle to BA′C, and let A′ move along this circle. Then A′BC + A′CB − A′ = constant.

$\therefore$ π − ABC + π − ACB − A = constant.

$\therefore$ ABC + ACB + BAC = constant.

$\therefore$ the area of ABC is constant; and A moves along the figure antipodal to the circle circumscribed to BA′C, that is, an equal and parallel circle.

Part of the circle is not included in the locus; for if tangents be drawn from C and D (fig. 3) to the small circle A′B′AB meeting the circle in C′ and D′, one at least of the lines joining any point between C′ and D′ to C and D must cut the circle in another point. Hence C′D′ is excluded from the locus. Since C′ and D′ are points antipodal to C and D, C′D′ = CD.

This is also evident from the second method of proving Lexell's theorem, since the loci for a number of triangles of different areason the same base are a number of circles all passing through the two points antipodal to the extremities of the common base.

§ 17. All triangles formed with CD as base and vertex in C′D′ are equal; and one of these triangles, together with one of the other set of equal triangles, forms half of the surface of the sphere.

Let BAC and BDC (fig. 5) be two triangles on the same base BC, but on opposite sides of it, and inscribed in the same circle.

Then $BAC + BDC = ABC + ACB + DBC + DCB$.

Now let AB and AC be produced to form a triangle, and also DB and DC; and let the angles of these triangles be A, B, C and A′, B′, C′ respectively.

Then from the above equality—

$$A + A' = \pi - B + \pi - C + \pi - B' + \pi - C';$$

$$\therefore A + B + C - \pi + A' + B' + C' - \pi = 2\pi = \tfrac{1}{2} \text{ surface of sphere};$$

and these two triangles have their vertices on the circle antipodal to the circle BACD. (The two triangles are on opposite sides of the sphere).

§ 18. The polar of Lexell's Theorem. If one angle of a triangle be fixed in position, and the sum of the sides containing this angle be constant, the side opposite the fixed angle will envelope a circle.

§ 19. If two sides of a triangle be given, the area is a maximum when the angle contained by the two given sides is equal to the sum of the other two. (Fig. 6.)

Let the side AB be supposed fixed, and the triangle to vary by change of the position of AC, the other given side.

Then the locus of the vertices of triangles of given area is a circle whose centre lies on OO′, the perpendicular bisector of AB; and the area will be greater the further the circle is from AB. Hence for a given length of AC the area is greatest when AC produced passes through the centre of the circle, as in the figure.

Let OO′ meet the circle in D; CA meet OO′ in E; produce DA and DB to meet at D′; and let O be the centre of the circle circumscribed to AD′B. Hence O′A passes through O.

Again $OA = O'D$; $\therefore \angle ODA = \angle O'AD$.

But since $\triangle ADB = \triangle ACB$,

$$\therefore ADB + DAB + ABD = ACB + CAB + ABC;$$

$$\therefore \quad 2(DAE + EDA) = ACB + DAC + DAE;$$

$$\therefore \quad 2(DAE + DAC) = ACB + DAC + DAE;$$

$$\therefore EAC = ECA. \quad \therefore BAC = ABC + BCA.$$

§ 20. The perpendicular bisectors of the sides and the bisectors of the angles of a triangle are concurrent.

§ 21. The internal points of bisection of any two sides of a triangle, and the external point of bisection of the remaining side (and also the three external points of bisection), are collinear.

This is the polar of the theorem that two external and one internal bisector of the angles of a triangle are concurrent.

A direct proof may also be given.

§ 22. The perpendiculars from the vertices of a triangle on the opposite sides are concurrent. (Fig. 7).

Let ABC be the triangle; AD, CF perpendicular to BC, AB. Draw AB′, CB′ perpendicular to AD, CF.
Make AC′=AB′; CA′=CB′; and join C′A′; bisect C′A′ in E.

Then, since A, C, E, are the middle points of the sides of A′B′C′, CE, if produced, will meet B′C′ in its external point of bisection, that is, in the pole of the line ADD′. ∴ CD′ is perpendicular to AD; ∴ C is the pole of ADD′; ∴ CA is a quadrant.

Hence, if CA be not a quadrant, E must coincide with B; and as CA may be any one of the sides, it is always possible to form a triangle such that A, B, C shall be the middle points of its sides, unless the sides of the triangle ABC be all quadrants. Now, the perpendiculars of the triangle ABC are the perpendicular bisectors of the sides of A′B′C′, and are therefore concurrent. If two sides, BA and BC say, are quadrants, B is the pole of AC; and since any line from B is, in that case, perpendicular to AC, it is not necessarily perpendicular to A′C′, and the theorem does not hold.

Cor.—The points of intersection of CA and C′A′, AB and A′B′, BC and B′C′, are collinear.

The theorem of § 22 may also be stated as follows:—In a complete quadrangle, if two diagonal angles be right angles, the third must also be a right angle.

§ 23. If two diagonals of a complete quadrilateral be quadrants, the third must also be a quadrant.

This is the polar of the theorem of § 22, according to the second statement of that theorem.

The following direct proof may also be given.

Let ACKH (fig. 8) be the quadrilateral, AK and CH being quadrants. Draw the perpendiculars of the triangle ABC.

Then O is the pole of HKL, and BO is perpendicular to AL;
∴ L is the pole of BO; ∴ BL is a quadrant.

§ 24. The perpendiculars from the vertices on the opposite sides of a triangle bisect the angles of the triangle formed by joining the feet of the perpendiculars. (Proof by means of the polar figure).

Let ABC be the polar triangle, L, M, N, the poles of the perpendiculars in the original triangle, *i.e.*, L is a point in BC such that LA is a quadrant, &c. Then DEF is the polar of the triangle formed by joining the feet of the perpendiculars in the original triangle; and it is required to show that L, M, N bisect the sides of DEF externally.

L, M, N are the poles of the perpendiculars of ABC;
∴ A, B, C are the middle points of the sides of DEF (§ 22).
∴ L, M, N bisect the sides externally.

Although not strictly within the scope of this paper, the following proof of the theorem of § 23 may be interesting.

Let ABCD (fig. 10) be the quadrilateral, AC and BD being quadrants. Then (AGCK) = − 1, and AC is a quadrant; ∴ GC = CK. Similarly, GB = BL.

Now, in the triangle LGK, B bisects LG internally, and A bisects GK externally; ∴ E bisects LK. And from triangle GLK F bisects LK externally; ∴ EF is a quadrant.

Note on the Condensation of a Special Continuant.

By Thomas Muir, M.A., F.R.S.E.

[Held over from Third Meeting.]

§ 1. The continuant referred to is that in which the elements of the main diagonal are all equal (to x, say), the elements of the one minor diagonal all equal (to b, say), and the elements of the other minor diagonal all equal (to c, say). It may be denoted by $F(b, x, c, n)$ when it is of the nth order. Professor Wolstenholme has recently given two elegant theorems regarding the condensation of $F(1, x, 1, n)$. I wish to establish the analogous theorems for $F(b, x, c, n)$.

§ 2. It may be necessary to premise that a determinant whose elements are all zeros, except those in the main diagonal and in the two diagonals drawn through the places (1, 3), (3, 1) parallel to the main diagonal, is expressible as the product of two continuants. Thus

$$\begin{vmatrix} a_1 & 0 & c_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & c_2 & 0 & 0 & 0 \\ b_1 & 0 & a_3 & 0 & c_3 & 0 & 0 \\ 0 & b_2 & 0 & a_4 & 0 & c_4 & 0 \\ 0 & 0 & b_3 & 0 & a_5 & 0 & c_5 \\ 0 & 0 & 0 & b_4 & 0 & a_6 & 0 \\ 0 & 0 & 0 & 0 & b_5 & 0 & a_7 \end{vmatrix} \text{ or } D_7,$$

$$= \begin{vmatrix} a_1 & c_1 & 0 & 0 \\ b_1 & a_3 & c_3 & 0 \\ 0 & b_3 & a_5 & c_5 \\ 0 & 0 & b_5 & a_7 \end{vmatrix} \cdot \begin{vmatrix} a_2 & c_2 & 0 \\ b_2 & a_4 & c_4 \\ 0 & b_4 & a_6 \end{vmatrix},$$

$$= K\begin{pmatrix} & -b_1c_1 & & -b_3c_3 & & -b_5c_5 & \\ a_1 & & a_3 & & a_5 & & a_7 \end{pmatrix} K\begin{pmatrix} & -b_2c_2 & & -b_4c_4 & \\ a_2 & & a_4 & & a_6 \end{pmatrix} \quad \text{....(I.)}$$

and similarly—

$$D_6 = \begin{vmatrix} a_1 & c_1 & 0 \\ b_1 & a_3 & c_3 \\ 0 & b_3 & a_5 \end{vmatrix} \cdot \begin{vmatrix} a_2 & c_2 & 0 \\ b_2 & a_4 & c_4 \\ 0 & b_4 & a_6 \end{vmatrix},$$

$$= K\begin{pmatrix} & -b_1c_1 & & -b_3c_3 & \\ a_1 & & a_3 & & a_5 \end{pmatrix} \cdot K\begin{pmatrix} & -b_2c_2 & & -b_4c_4 & \\ a_2 & & a_4 & & a_6 \end{pmatrix} \quad \text{....(II.)}$$

and therefore, as we may observe in passing,

$$\frac{D_7}{D_6} = a_7 - \cfrac{b_5c_5}{a_5 - \cfrac{b_3c_3}{a_3 - \cfrac{b_1c_1}{a_1}}} \quad \text{...(III.)}$$

§ 3. Also, we may note that since

$$K\begin{pmatrix} & -b^2 & & -b^2 & & -b^2 & & -b^2 & \\ a+b & & a & & a & & a & & a+b \end{pmatrix} =$$

$$K\begin{pmatrix} & -b^2 & & -b^2 & & -b^2 & & -b^2 & \\ a & & a & & a & & a & & a+b \end{pmatrix} + bK\begin{pmatrix} & -b^2 & & -b^2 & & -b^2 & \\ a & & a & & a & & a+b \end{pmatrix}$$

and since the first term in the right-hand member equals

$$(a+b)K\begin{pmatrix} & -b^2 & & -b^2 & & -b^2 & \\ a & & a & & a & & a \end{pmatrix} - b^2K\begin{pmatrix} & -b^2 & & -b^2 & \\ a & & a & & a \end{pmatrix},$$

and the second term equals

$$b \cdot K\begin{pmatrix} & -b^2 & & -b^2 & & -b^2 & \\ a & & a & & a & & a \end{pmatrix} + b^2K\begin{pmatrix} & -b^2 & & -b^2 & \\ a & & a & & a \end{pmatrix},$$

we have the identity in continuants

$$K\begin{pmatrix} & -b^2 & & -b^2 & & -b^2 & & -b^2 & \\ a+b & & a & & a & & a & & a+b \end{pmatrix} = (a+2b)K\begin{pmatrix} & -b^2 & & -b^2 & & -b^2 & \\ a & & a & & a & & a \end{pmatrix} \quad \text{(IV.)}$$

§ 4. Now, taking the case of F (b, x, c, n) where n is odd, we have

$$F(b, x, c, 7) = \begin{vmatrix} x & b & \cdot & \cdot & \cdot & \cdot & \cdot \\ c & x & b & \cdot & \cdot & \cdot & \cdot \\ \cdot & c & x & b & \cdot & \cdot & \cdot \\ \cdot & \cdot & c & x & b & \cdot & \cdot \\ \cdot & \cdot & \cdot & c & x & b & \cdot \\ \cdot & \cdot & \cdot & \cdot & c & x & b \\ \cdot & \cdot & \cdot & \cdot & \cdot & c & x \end{vmatrix} = \begin{vmatrix} x-c & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -b & x-c & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -b & x-c & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -b & x-c & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -b & x-c & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -b & x-c \\ \cdot & \cdot & \cdot & \cdot & \cdot & -b & x \end{vmatrix}$$

and therefore by multiplication

$$\left[F(b,x,c,7)\right]^2 = \begin{vmatrix} x^2-bc & 0 & -b^2 & \cdot & \cdot & \cdot & \cdot \\ 0 & x^2-2bc & 0 & -b^2 & \cdot & \cdot & \cdot \\ -c^2 & 0 & x^2-2bc & 0 & -b^2 & \cdot & \cdot \\ \cdot & -c^2 & 0 & x^2-2bc & 0 & -b^2 & \cdot \\ \cdot & \cdot & -c^2 & 0 & x^2-2bc & 0 & -b^2 \\ \cdot & \cdot & \cdot & -c^2 & 0 & x^2-2bc & 0 \\ \cdot & \cdot & \cdot & \cdot & -c^2 & 0 & x^2-bc \end{vmatrix}$$

$$= \begin{vmatrix} x^2-bc & -b^2 & \cdot & \cdot \\ -c^2 & x^2-2bc & -b^2 & \cdot \\ \cdot & -c^2 & x^2-2bc & -b^2 \\ \cdot & \cdot & -c^2 & x^2-bc \end{vmatrix} \begin{vmatrix} x^2-2bc & -b^2 & \cdot \\ -c^2 & x^2-2bc & -b^2 \\ \cdot & -c^2 & x^2-2bc \end{vmatrix} \text{ by § 2.}$$

$$= x^2 \begin{vmatrix} x^2-2bc & -b^2 & \cdot \\ -c^2 & x^2-2bc & -b \\ \cdot & -c^2 & x^2-2bc \end{vmatrix}^2 \text{ by § 3.}$$

and consequently we have

$$F(b, x, c, 7) = xF(b^2, x^2-2bc, c^2, 3),$$

the general theorem evidently being

$$F(b, x, c, 2n+1) = xF(b^2, x^2-2bc, c^2, n) \qquad \text{....(V.)}$$

In exactly the same way we find the complementary theorem

$$F(b, x, c, 2n) = F(b^2, x^2-2bc, c^2, n) + bcF(b^2, x^2-2bc, c^2, n-1) \ldots \text{(VI.)}$$

Beechcroft, Bishopton,
2nd Jan. 1884.

Pascal's Essais pour les Coniques.

By W. J. MACDONALD, M.A.

In 1640, when only 16 years of age, Pascal published a tract of a few pages with the above title. It contains only a few enunciations, and concludes with the statement that the author has several other theorems and problems, but that his inexperience, and the distrust he has of his own powers, do not allow him to publish them till they have been examined by competent judges. He afterwards wrote a complete work (*opus completum*) on the Conics, which was submitted to Leibnitz by M. Périer, Pascal's brother-in-law. Leibnitz recommended that it should be published; but this was not done, and we know its contents only from the analysis which Leibnitz sent back to M. Périer.

One feature which distinguishes Modern Geometry from the Ancient Geometry, is that a few propositions of great generality are proved, and from these a large number of others are deduced as corollaries. Now, Pascal's work presents this feature in a marked degree; for he takes up a single proposition—the well-known "Mystic Hexagram," as he called it—and from it deduces all his others, four hundred corollaries, we are told. It has been suggested that the proposition is really due to Desargues; but he himself speaks of a proposition as Pascal's which can be none other than this.

I propose now to give an account of the contents of the earlier work, modernizing the enunciations, and supplying demonstrations on the lines on which I imagine Pascal himself worked.

We have first a definition equivalent to that of *concurrent lines* (parallel lines are included), and then a *conic* is defined to include the *circle*, *parabola*, *ellipse*, *hyperbola*, and *a pair of intersecting lines*.

Then comes Lemma I, which is the Hexagram for the circle: If a hexagon* be inscribed in a circle the three points in which the pairs of opposite sides intersect are collinear. (Fig. 18)

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* By a hexagon is to be understood the figure formed by joining consecutively any six points on the circumference of the circle. Sixty different figures are possible according to the order in which the points are joined.

In ΔPQR, with transversal* BC, $\frac{PC}{CQ} \cdot \frac{QY}{YR} \cdot \frac{RB}{BP} = -1$;

" " " DE, $\frac{PD}{DQ} \cdot \frac{QE}{ER} \cdot \frac{RX}{XP} = -1$;

" " " FA, $\frac{PZ}{ZQ} \cdot \frac{QF}{FR} \cdot \frac{RA}{AP} = -1$;

$$\therefore \quad \underbrace{\frac{PZ}{ZQ} \cdot \frac{QY}{YR} \cdot \frac{RX}{XP}}_{\alpha} \cdot \underbrace{\frac{PC \,.\, PD \,.\, QE \,.\, QF \,.\, RA \,.\, RB}{PA \,.\, PB \,.\, QC \,.\, QD \,.\, RE \,.\, RF}}_{\beta} = -1$$

But $\beta = +1$ (Eucl. III. 36); $\therefore \quad \alpha = -1$;

$\therefore$ X, Y, Z are collinear.

This proof holds for every disposition of the points A,B,C,D,E,F, provided they are joined in alphabetical order.

Lemma II. states that lines which are concurrent in any plane are concurrent when projected into any other. From this follows at once Lemma III., in which the circle of Lemma I. is replaced by any conic. Conversely also: If the intersections of the pairs of opposite sides of a hexagon are collinear, the vertices of the hexagon lie on a conic.

Since a conic is determined by five points, and this proposition gives us a relation between these and any sixth point on the conic, it is clear that the proposition is one of great generality, and admits of many corollaries, some of which I shall now indicate.

It enables us to construct a conic, five points on it being given.

Let 1, 2, 3, 4, and 5 be the given points.

Through 1 draw any line meeting 34 in X; join 12 and 45 meeting in Y; join 23 and XY meeting in Z; join Z5 meeting X1 in 6.

Then 6 is a point in the conic.

By drawing different lines through 1 any number of points on the conic can thus be determined.

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* Both Pascal and Desargues appear to have made much use of the propositions known as Menelaus' and Ceva's Theorems (Fig. 17),

that if $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = -1$,

D, E, and F are collinear; and conversely;

and if $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$,

AD, BE, and CF are concurrent; and conversely;

and these I shall assume as known.

Let 1 and 6 coincide, then 16 becomes the tangent at that point. (Fig. 19.)

This gives the proposition : If a pentagon* be inscribed in a conic, the points of intersection of the first and fourth sides, and of the second and fifth sides, are collinear with the point in which the third side meets the tangent at the opposite vertex.

From this we obtain the solution of the problem : To draw a tangent to a conic from a point on it, by the ruler alone.

Take any other points 2, 3, 4, and 5 on the conic. Join 12 and 45 meeting in X ; join 23 and 51 meeting in Y ; join XY and 43 meeting in Z.

Join Z1, which is the required tangent.

The problem in construction which this case solves is : To construct a conic, having given three points on the conic and a tangent with its point of contact. The construction is similar to that given for five points.

The hexagon may be reduced to a quadrilateral in two ways :

1. By considering two adjacent vertices of the quadrilateral each to contain two consecutive vertices of the hexagon.

2. By so considering two opposite vertices.

These give the two propositions :

1. The tangents at two adjacent vertices of a quadrilateral inscribed in a conic, meet on the line joining the intersection of the diagonals with the intersection of the pair of opposite sides which pass through the vertices.

2. The tangents at two opposite vertices of a quadrilateral inscribed in a conic, meet on the line joining the intersections of pairs of opposite sides.

[These are really the same propositions, and differ only in the order in which the points are supposed to be joined.]

The problem corresponding to this case is : Given a pair of tangents with their points of contact, and any other point on the conic, to construct it.

This case implicitly contains the Theory of Pole and Polar ; and it is the opinion of both Chasles and Poncelet that Pascal had developed the equivalent of that Theory in Book III. of the *Opus Completum.* The title of that Book is *De quatuor tangentibus, et rectis*

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* Pentagon here has the same extended meaning as hexagon,

puncta tactuum jungentibus, unde rectarum harmonice sectarum et diametrorum proprietates oriuntur; and Leibnitz expressly states that it was founded on the properties of the hexagram.

If now three pairs of points coincide we have this proposition: The points in which the sides of a triangle inscribed in a conic meet the tangents at the opposite vertices are collinear.

Lemma III. also implicitly contains Carnot's Theorem. For if X,Y,Z be collinear, (Fig. 18)

$$\frac{PZ}{ZQ} \cdot \frac{QY}{YR} \cdot \frac{RX}{XP} = -1;$$

$$\therefore \quad \frac{PC \cdot CD \cdot QE \cdot EF \cdot RA \cdot RB}{PA \cdot PB \cdot QC \cdot QD \cdot RE \cdot RF} = 1,$$

which is Carnot's Theorem.

This is another proposition of great generality concerning points on a conic. Pascal apparently knew it, for, as we shall see immediately, he extended it to eight points.

To return to the *Essais.*

In Fig. 20 he says

$$\frac{PM}{MA} \cdot \frac{AS}{SQ} = \frac{PL}{LA} \cdot \frac{AT}{TQ}.$$

Because PKNOVQ is a hexagon, QP, ON, and MS are concurrent in X_1 (say).

Because PKONVQ is a hexagon, QP, ON, and LT are concurrent in X_2 (say).

But two lines in each set are the same;

$\therefore$ X_1 and X_2 are the same point (say) X.

Hence in ΔAPQ,

with transversal SMX, $\dfrac{PM}{MA} \cdot \dfrac{AS}{SQ} \cdot \dfrac{QX}{XP} = -1;$

with transversal TLX, $\dfrac{PL}{LA} \cdot \dfrac{AT}{TQ} \cdot \dfrac{QX}{XP} = -1;$

$$\therefore \quad \frac{PM}{MA} \cdot \frac{AS}{SQ} = \frac{PL}{LA} \cdot \frac{AT}{TQ}.$$

The next proposition (Fig. 21) is equivalent to this: If from a point there be drawn three lines to cut the sides of an angle the anharmonic ratio of the segments made on one side is equal to that of the segments made on the other.

In Δ ABE, with the transversals DH and CH,

$$\frac{AD}{DB}\cdot\frac{BH}{HE}\cdot\frac{EG}{GA}=-1=\frac{AC}{CB}\cdot\frac{BH}{HE}\cdot\frac{EF}{FA};$$

$$\therefore \qquad \frac{AD.BC}{AC.BD}=\frac{AG.EF}{AF.EG}.$$

Then follows the extension of Carnot's Theorem to the quadrilateral, to which we have already referred.

If (Fig. 22) the sides of the quadrilateral ACLH be cut by a conic,

AB.AE.CP.CR.HF.HK.LM.LO = AF.AK.CB.CE.HO.HM.LR.LP.

Apply Carnot's Theorem successively to the Δs ACG and LHG, and we have

1. AB.AE.CP.CR.GF.GK = AF.AK.CB.CE.GP.GR;
2. LM.LO.HF.HK.GP.GR = LR.LP.GF.GK.HM.HO.

From the multiplication of these the proposition follows.

A particular case of this theorem gives us an important property of conics, that if through any point a pair of secants to a conic be drawn parallel to two fixed directions, the rectangles under their segments are in a fixed ratio.

For if A and L are at infinity, $\dfrac{CP.CR}{CB.CE}=\dfrac{HO.HM}{HF.HK}$.

Next we have Desargues' proposition: That any transversal is cut by a conic, and the sides of an inscribed quadrilateral in six points which are in involution.*

In Fig. 23, apply Carnot's Theorem to ΔAA'F and the conic; then AB.AB'.A'L.A'M.FR.FS = AS.AR.FM.FL.A'B'.A'B.

Apply the same Theorem to ΔAA'F and the pair of lines LS, RM; then AS.AR.FM.FL.A'C'.A'C = AC.AC'.A'L.A'M.FR.FS.

Multiply and suppress common factors;

$$\therefore \qquad \frac{AB.AB'}{A'B'.A'B}=\frac{AC.AC'}{A'C'.A'C}.$$

He then suggests one or two problems, among which is: To draw a pair of tangents to a conic from an external point. His solution was, no doubt, that which depends on the polar properties of the complete quadrilateral, as we have seen that he probably knew these.

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* Six points A, A', B, B', C, C' are in involution, if $\dfrac{AB.AB'}{A'B'.A'B}=\dfrac{AC.AC'}{A'C'.A'C}$.

In a communication dated 1654, and addressed to a society of savants, which in 1666 became the Academy of Sciences, Pascal stated that he had written a complete treatise on the conics, founded mainly on a single proposition. This work, as we have already stated, was, after its author's death, sent for examination to Leibnitz; and though it has been lost, we have the analysis of it which Leibnitz made for M. Périer, Pascal's brother-in-law. In spirit and method it anticipates the Modern Geometry of our century, and entitles Pascal to the credit of having been one of its founders.

Sixth Meeting, April 10th, 1884.

THOMAS MUIR, Esq., M.A., F.R.S.E., President, in the Chair.

On the Teaching of Elementary Geometry.

By A. J. G. BARCLAY, M.A.

[*Abstract.*]

This paper was prepared at the suggestion of the committee as the first of a series on the teaching of elementary mathematics, in the belief that an occasional paper of this nature, with discussions, would be useful.

In the introduction it was suggested that, as secondary education in this country was apparently on the eve of considerable changes, the present was an opportune time for discussing the whole subject of school mathematics; and also that the Society should be prepared to form a scheme of a mathematical course for both teaching and examination purposes.

The following points were specially referred to:

(1) That the most suitable time for a pupil to commence geometry is about the age of twelve. (2) That the introduction to the subject should be made with the usual definitions, along with numerous exercises in the making and naming of figures; this, rather than the course of geometrical drawing, unaccompanied by definitions, suggested by the Society for the Improvement of Geometrical Teach-

ing. (3) That propositions should not be repeated in a rigid form of words, but that the teacher insist on intelligent expression. (4) That geometry is a subject eminently fitted for oral exposition; and that each proposition, before being prescribed to be learned, ought to be taught to the class. (5) That the text book should contain the propositions put as clearly as possible with easy exercises accompanying. (6) That symbols and contractions, as far as their use tends to simplicity, should be employed. (7) That the work be systematically reproduced in writing. (8) That revision might occasionally be made by retracing the chain of propositions. (9) That the quality of the geometrical work done, rather than its quantity, determines its educational value.

On Voting.

By A. MACFARLANE, D.Sc., F.R.S.E.

Suppose that we have c candidates, e electors, s seats, v votes.

There are at least three different kinds of voting to consider: Simple, Combinational, and Cumulative.

I. *Simple voting.* By simple voting I mean any case in which an elector has only one vote. Denote the candidates by A, B, C, D. The possible ways in which elector No. 1 can vote are given by

$$A_1 + B_1 + C_1 + D_1;$$

similarly for elector No. 2,

$$A_2 + B_2 + C_2 + D_2.$$

The possible results of No. 1 and No. 2 voting are obtained by multiplying together the possible ways for each, hence they are :—

$$\begin{array}{llll} A_2A_1 + A_2B_1 + A_2C_1 + A_2D_1 \\ B_2A_1 + B_2B_1 + B_2C_1 + B_2D_1 \\ C_2A_1 + C_2B_1 + C_2C_1 + C_2D_1 \\ D_2A_1 + D_2B_1 + D_2C_1 + D_2D_1. \end{array}$$

It will be observed that along one diagonal we have the cases in which the two electors vote for the same candidate. If it is considered inessential from whom the vote comes, then the ways to the left of the diagonal are duplicates of the ways to the right.

When there is a third elector, we have to multiply the result by

$$A_3 + B_3 + C_3 + D_3.$$

From the mode of derivation, it is evident that the number of different ways in which the voting may result is c^e.

But if it is considered immaterial from whom the vote comes, the suffixes may be dispensed with, and the different ways are the homogeneous products of the symbols A, B, C, D. Hence the number of different ways is

$$\frac{\overline{c+e-1}\,!}{e!\;\overline{c-1}\,!}.$$

II. *Combination voting.* By combination voting I mean voting in which one elector has more than one vote, but must choose a combination. Suppose that the combination is of two, out of A,B,C,D. Then for the 1st elector

$$A_1B_1 + A_1C_1 + A_1D_1 + B_1C_1 + B_1D_1 + C_1D_1,$$

and for the 2nd elector

$$A_2B_2 + A_2C_2 + A_2D_2 + B_2C_2 + B_2D_2 + C_2D_2\,;$$

hence the different results of the two electors voting are

$$\begin{array}{l}
A_2A_1B_2B_1 + A_2A_1B_2C_1 + A_2A_1B_2D_1 + A_2B_1B_2C_1 + A_2B_1B_2D_1 + A_2B_2C_1D_1 \\
A_2A_1B_1C_2 + A_2A_1C_1C_2 + A_2A_1C_2D_1 + A_2B_1C_1C_2 + A_2B_1C_2D_1 + A_2C_1C_2D_1 \\
A_2A_1B_1D_2 + A_2A_1C_1D_2 + A_2A_1D_2D_1 + A_2B_1C_1D_2 + A_2B_1D_2D_1 + A_2C_1D_2D_1 \\
A_1B_2B_2C_2 + A_1B_2C_2C_1 + A_1B_2C_2D_1 + B_2B_1C_2C_1 + B_2B_1C_2D_1 + B_2C_2C_1D_1 \\
A_1B_2B_1D_2 + A_1B_2C_1D_2 + A_1B_2D_2D_1 + B_2B_1C_1D_2 + B_2B_1D_2D_1 + B_2C_1D_2D_2 \\
A_1B_1C_2D_2 + A_1C_1C_2D_2 + A_1C_2D_1D_2 + B_1C_1C_2D_2 + B_1C_2D_2D_1 + C_2C_1D_2D_1
\end{array}$$

The number of possible ways in which the combination may be chosen is

$$\frac{\overline{c-s+1}\,!}{\overline{c-1}\,!\;s!};$$

hence the number of different ways in which the voting may take place is

$$\left\{\frac{\overline{c-s+1}\,!}{\overline{c-1}\,!\;s!}\right\}^e.$$

If it is considered immaterial from whom the vote comes, then all the terms on one side of the diagonal are cut off as before, but in addition some of the terms on the other side of the diagonal. For example, in the case above, there are two terms in the other diagonal which have to be cut out. There is evidently an expression for the number, but it is pretty complex.

III. *Cumulative voting.* In cumulative voting an elector has a plurality of votes, and he is not obliged to choose a combination; his votes are independent of one another. Hence the different ways in which he can vote are represented by the homogeneous products.

He may give any number of votes up to s, the number of seats, hence the number of different ways in which he can vote is

$$\frac{c+s\,!}{s!\,c!}.$$

When there are e electors voting in this way, the total number of ways (states of the poll) is the same as if one elector had es cumulative votes. Hence

$$\frac{c+es\,!}{es!\,c!}.$$

Mr J. S. Mackay gave the following solution of Mr Edward's problem, (see p. 5):

Between two sides of a triangle to inflect a straight line which shall be equal to each of the segments of the sides between it and the base.

Let ABC (fig. 15), be a triangle, and let the side AB be less than AC. Draw any straight line DE parallel to BC, and cutting the sides AB, AC, or AB, AC produced either below the base or through the vertex, in D and E. Cut off CF′ equal to BD; with centre F′ and radius CF′ cut DE or DE produced at the points G′; and join F′G′. Let CG′ meet AB or AB produced at G, and draw GF parallel to G′F′. GF is the line required.

For through G′ draw A′B′ parallel to AB, and meeting the sides AC, BC, or AC, BC produced, in A′, B′.

Then B′G′ = BD = CF′ = F′G′.

Now, since the quadrilaterals CB′G′F′, CBGF are similar, and either similarly or oppositely situated, C being their centre of similitude; and since B′G′ = G′F′ = F′C;
therefore BG = GF = FC.

Seventh Meeting, May 9th, 1884.

THOMAS MUIR, Esq., LL.D., F.R.S.E., President, in the chair.

The Hypothesis of Le Bel and Van 't Hoff.

By Professor A. CRUM BROWN, University of Edinburgh.

Arago observed in 1811 that if a plane polarised ray of light be passed vertically through a plate of quartz cut at right angles to the crystallographic axis, the ray emerges plane polarised but with its plane of polarisation inclined at an angle to the original plane of polarisation, this angle (or the amount of rotation of the plane of polarisation about the direction of the ray) being proportional to the thickness of the plate. Biot further showed that in some quartz crystals this rotation is in one sense, in others in the opposite. In 1821 Herschel proved that the sense of this rotation was connected with the inclination of the so-called plagiedral faces to the faces of the prism. In 1830 Naumann gave a very complete account of the crystallography of quartz, showed that the two kinds of crystals are mirror images of each other, and gave to this relation the name of Enantiomorphism. Quartz long remained the only known solid crystalline substance having the property of rotating the plane of polarisation in the way above described. In 1853 Rammelsberg discovered that the crystals of sodium chlorate are enantiomorph, and next year Marbach showed that they rotate the plane of polarisation, and that the two sets of enantiomorph crystals rotate the plane in opposite senses, so that, like quartz, sodium chlorate is enantiomorph optically as well as crystallographically.

The same relation has since been observed in some other substances.

All these substances show optical enantiomorphism while they are in the solid, crystalline state. But if they are fused or dissolved, the liquid has no rotating action on the plane of polarisation.

It is interesting to notice that these crystals are all either regular (sodium chlorate, sodium bromate) or uniaxial (quartz). But there

are many substances, whose *solutions* possess optical activity, as the property of rotating the plane of polarisation is called. In them this property must therefore depend on the structure of the molecules, and not on the way in which these are arranged in the crystal.

Pasteur in an elaborate series of papers worked out the enantiomorphic relations especially in the case of tartaric acid. He showed that besides the ordinary tartaric acid (the solution of which rotates the plane of polarisation to the right) there is its enantiomorph—left-handed tartaric acid, that racemic acid is an optically inactive compound of these two opposite tartaric acids, and that there is a fourth form which is also optically inactive, but which cannot be separated as racemic acid can, into a right and a left-handed component. It is interesting to notice that in this and in all other cases where optical activity is observed in the liquid or dissolved state, the crystals of the substance (if it can be obtained in crystals) are biaxial, while as already stated optically active *crystals* are regular or uniaxial—in other words crystallographic enantiomorphism depends in the latter on tetartohedry, in the former on hemihedry.

It is obvious that optical activity in the dissolved state, that is optical activity of the molecules, must depend on the structure of the molecules, that is, on the chemical constitution of the substance.

The hypothesis as to the relative position of the atoms in the molecule of an organic compound, published nearly simultaneously by Le Bel in Paris, and by Van 't Hoff in Rotterdam, gives a plausible explanation of this relation. I shall state the main points of the theory without adhering closely to the order in which either of its authors has developed it.

An atom of carbon can combine with four atoms of hydrogen, or four atoms of chlorine, or four atoms of any so-called *monad* element or compound radical. It can also combine with several different monad atoms or compound radicals in such a manner that the sum of them is four. So that $C\alpha\beta\gamma\delta$ may be taken to represent a compound of carbon where $\alpha,\beta,\gamma,\delta$ stand each for a monad atom or compound radical. When α,β,γ and δ are all different the carbon atom is said to be asymmetric. Now, all subtances optically active in solution (or in the liquid state) whose chemical constitution is known contain an asymmetric carbon atom. The converse of this is not true—we know substances with an asymmetric carbon atom which

have not been observed to possess any action on the plane of polarisation. Now, if we suppose the four atoms (leaving, for the meantime, compound radicals out of consideration) α,β,γ and δ to be situated around the atom of carbon, the simplest supposition is that if they are all of the same kind (that is if the compound be $C\alpha\alpha\alpha\alpha$) the carbon atom C will occupy the centre of figure of a regular tetrahedron while the atoms $\alpha,\alpha,\alpha,\alpha$ combined with it occupy the apices. Now, if all the atoms combined with the carbon atom are not of the same kind, we may reasonably assume that the tetrahedron, at the apices of which these atoms are, will not be a regular tetrahedron, because each atom will have its own appropriate distance from C. As long, however, as they are not *all* different there will be only one figure; but if they are all different there will be two possible figures with $\alpha,\beta,\gamma,\delta$ at the apices and each of them at its appropriate distance from C, and these two figures will be enantiomorph. That this is so will be at once seen if we consider any one face of the tetrahedron—say that which has α,β,γ, at its corners. Looking at this face from the outside—that is, with δ further from us than the face in question—we may have the order

$$\begin{array}{ccccccc} & \beta & & & & \beta & \\ \alpha & & \gamma & \text{or} & \gamma & & \alpha \end{array},$$

the one being the mirror image of the other.

On this supposition as to the relative position of the atoms, a compound containing an asymmetric carbon atom can exist in one or other of two forms precisely similar in every respect, except that they are enantiomorph.

And we can easily see why substances which have an asymmetric carbon atom are not always optically active. In any ordinary way of making such a substance it is plainly as likely that the one form should be produced as the other. Therefore, as the number of molecules in any quantity of the substance that we can deal with is practically infinite, the ratio of the number of the one kind to that of the other kind will be practically unity, and, therefore, the rotatory effects will precisely balance one another. So that we cannot expect an optically active substance to be produced from optically inactive materials without the intervention of some agent which can act differently on the two enantiomorphs, and enable us to obtain one or both of them separately out of the mixture. Of such agents we have several kinds.

(1) Crystallisation. The two enantiomorphs may crystallise in identical forms, but usually there are present certain faces on one side of the one and on the other side of the other, distinguishing them. In such cases, if the two enantiomorphs do not unite together, we can pick out the two sorts of crystals and thus separate the two substances. Further, if we prepare a supersaturated solution of one of the enantiomorphs, we find (in some cases, at all events) that crystallisation is caused by the addition of a crystal of the same kind, but not of the other kind; so that if we prepare a supersaturated solution of the mixture and drop into it a crystal of the right-handed sort, only the right-handed substance will crystallise out. In this way, separation has been effected by dropping simultaneously into such a supersaturated solution, at different parts of the vessel containing it, two crystals, one of the one sort the other of the other, when each substance crystallised out separately at the place where the crystal of its own kind was placed.

(2) The action of another optically active substance. An example will make this mode of separation clear. Right-handed tartaric acid forms a definite crystalline compound with left-handed asparagine; left-handed tartaric acid with left-handed asparagine gives an uncrystallisable gummy substance. We might illustrate this by an analogy. Right-handed and left-handed men can both use tools of the sort that Professor Tait calls amphicheiral, such as chisels, knives, axes, and planes, but the case is different if you give them right or left-handed tools such as scissors or screws. The right-handed man with the left-handed tool is as awkward as the tartaric acid with the wrong kind of asparagine.

(3) Fermentation. Fermentation takes place in many solutions in the presence of growing fungi. The nature of the chemical change depends, of course, on the nature of the dissolved substance and also on the kind of fungus.

M. Le Bel found that some of the fungi act more readily on one than on the other of two enantiomorph substances. These fungi are, in fact, not indifferent or amphicheiral but pick out the one kind of molecules and cause their oxidation while leaving the other. This may give us some idea how it is that plants and animals often contain optically active substances.

Tartaric acid is peculiarly interesting in connection with this

theory, both because its optical, chemical, and crystallographical properties have been very fully examined, and also because it is an example of a special case. The constitution of tartaric acid is represented by the graphic formula

$$\begin{array}{ccccccccccccccc}
 & & & & & & H & & H & & & & & & \\
 & & & & & & | & & | & & & & & & \\
 & & & & O & & O & & O & & O & & & & \\
 & & & & \| & & | & & | & & \| & & & & \\
H & — & O & — & C & — & C & — & C & — & C & — & O & — & H \\
 & & & & & & | & & | & & & & & & \\
 & & & & & & H & & H & & & & & &
\end{array}$$

Here the two middle atoms of carbon are both asymmetric and precisely similar to one another in their relation to the rest of the compound. Here, then, we should expect the following forms:—(*a*) Right-handed, in which both asymmetric carbon atoms are right-handed. (*b*) Left-handed, in which both are left-handed. (*c*) Inactive, in which one is right and the other left, that is to say, the one is the mirror image of the other. (*d*) Inactive by compensation — a mixture (or compound) of *a* and *b*. Now, this is exactly what we have. No other case has been investigated in which there are two precisely similar asymmetric carbon atoms, and no other case is known where there is an inactive form besides the mixture (or compound) of the two enantiomorphs.

The theory also gives a plausible explanation of the existence of two different acids, maleic and fumaric, both having the constitution indicated by the graphic formula

$$\begin{array}{ccccccccccccccc}
 & & & & O & & H & & H & & O & & & & \\
 & & & & \| & & | & & | & & \| & & & & \\
H & — & O & — & C & — & C & = & C & — & C & — & O & — & H.
\end{array}$$

and yielding by the addition of bromine, two isomeric acids, the one corresponding to inactive tartaric acid, the other to racemic acid (the compound of the two active tartaric acids).

Eighth Meeting, June 13th, 1884.

A. J. G. Barclay, Esq., M.A., Vice-President, in the Chair.

On the Representation of the Physical Properties of Substances by means of Surfaces.

By W. Peddie.

If the physical state of a substance is completely defined when the simultaneous values of three of its properties are given, then, by measuring off along three rectangular axes, from any point chosen as origin, lengths proportional to these values, we determine a point which represents completely the physical state of the substance. And, evidently, each point lies on a surface, the equation to which is determined by the three co-ordinate properties. If, in the equation to the surface, we give one of the variables a definite value, we get the equation to a contour-line of the surface which represents the necessary relation subsisting between the remaining two properties when the other is constant.

The nature of any quantity is completely known when it is understood *what* units are involved in its measurement and *how* they are involved. Thus a speed involves the unit of length directly, and the unit of time inversely; an acceleration involves a length directly, and the square of a time inversely. When we are dealing with space, however, the unit of length alone is involved. We say that the space considered has one, two, or three, &c., *dimensions*, according as the unit is involved to the first, second, or third, &c., power. A line has only one dimension. Given a certain point on the line as origin, only one number, with the proper sign attached, is required to completely specify the relative position of any other point on the line. A surface has two dimensions. Two directed lengths are necessary to define the position of a point on it with reference to any other point taken as origin. Thus we speak of the position of a point on the earth's surface as being so much north or south of a certain line, and so much east or west of another. In the three-

dimensioned space to which we are accustomed, three such lengths are required. Thus we speak of the length, breadth, and thickness of a solid.

The intersection of any surface which has a constant characteristic with the surface of a solid of three dimensions is a contour-line. The analogue in two dimensions of a contour-line is what may be termed contour-points, that is, the points in which a line, along which some quantity is constant, cuts the boundary of a surface. The boundary of a surface is a line and exists in two-dimensional space; so that, in two-dimensional space, contours have no dimensions. Similarly, in three-dimensional space, contours are of one dimension. The properties of four-dimensional space, or even n-dimensional space, can be treated mathematically; but, from want of experience, it is impossible to imagine the nature of such space. Contours in it would be surfaces,—the surfaces of intersection of solids, throughout which some quantity was constant, with solids existing in four-dimensional space.

The contour-lines most widely known are those formed by the intersection of level surfaces with the surface of the earth. The line of sea-board is one such contour-line. The essential feature of these lines is that by means of them a third dimension is represented upon a surface. An ordinary map with numbers marked upon it indicating the heights of various places, represents roughly the third dimension. So also does a chart with numbers corresponding to the various depths of the sea. A line drawn free-hand through the points of equal height or depth would approximately coincide with a contour-line. We may obtain any number of contour-lines by supposing the sea-level to rise or fall as necessary. It must be specially observed that the surfaces intersecting the earth's surface are *level*. From this it follows that, since the earth is not spherical in shape, contour-lines are not lines of constant height above, say, ordinary sea-level taken as a standard. The assumption that they are lines of constant height will not introduce appreciable error, however, if the value of gravity is not sensibly different at different parts of the same line. The quantity which is constant over a level surface is the work required to be done in order to raise a given mass to it against gravity from any station on the standard level. This is, therefore, the quantity which is constant along the contour-line. Since the work so done is equal to the kinetic energy (the product of the mass

into half the square of the velocity acquired) which would be gained by the mass if allowed to slide from the upper to the lower level by any path, we may define the constant quantity, independently of the mass, as half the square of the velocity acquired by a body falling, by any path, from the upper level to the standard point on the lower level.

To determine the nature of the surface as indicated by peculiarities in the form of the contour-lines, let us suppose the earth to be entirely submerged so that we have only one region, and that a region of depression. If now we suppose the water to be slowly absorbed by the solid matter of the earth, regions of elevation will be formed gradually until, finally, we shall have only one region, and that a region of elevation. Before a region of elevation is formed we have a *summit* appearing above the water-level; and, when the water subsides out of a region of depression, we have a lowest-point, or *immit*, appearing. The number of regions of elevation and depression may vary in two ways. We may have two regions of elevation running into each other as the water sinks. The point where they first meet is termed a *pass*, (see Fig. 24; p_1, p_2, &c.). Again, a region of elevation may throw out arms which run into each other and so cut off a region of depression. The point where they first meet is termed a *bar*, (Fig. 24; b_1, b_2, &c.). The contour-line for a level immediately underneath that corresponding to the bar has a closed branch within the region of depression cut off. Thus the closed curve at I_4, Fig. 24, is part of the contour-line ux. If a chart of an insular high-land be constructed as above indicated, a pass occurs at the node (see Fig. 24) of a figure-of-eight curve, (or *out-loop* curve, as Professor Cayley has termed it); while a bar occurs at the node of an *in-loop* curve. If, in Fig. 24, we interchange the summits and immits, the passes and bars, we see that, in the chart of an island-basin (Maxwell, on Hills and Dales, Philosophical Magazine, series 4, vol. 40, Dec. 1870, p. 427), a pass is represented by the node of an *in-loop* curve, and a bar corresponds to the node of an *out-loop* curve. If there were any advantage in having passes and bars always indicated by the node of the same kind of curve respectively, this could be attained by affixing the positive sign not constantly to the region on the same side of the level surface, but to the region towards which or from which the surface was moving.

As a particular case, two regions of elevation may run into each

other at a number of points simultaneously. Of these points, one must be taken as a pass and the others as bars. We may have also singular points where, for example, three or more regions of elevation meet. Such points are termed *double*, *treble*, &c., passes. Similarly, we may have multiple bars.

Before a pass can be formed there must be two summits, and for every additional pass there is another summit. Thus the number of summits is one more than the number of passes. So also the number of immits is one more than the number of bars.

Slope-lines are lines drawn everywhere perpendicular to the contour-lines. Evidently the steepness of a district is indicated on a chart by the closeness of the contour-lines. There are two kinds of slope-lines, however, which are specially important. These are the slope-lines drawn from summits to passes or bars, and from passes or bars to immits. The first of these can never reach an immit, and are termed *water-sheds*. The second can never reach a summit, and are termed *water-courses*.

A perpendicular precipice is indicated on a chart by the running together of two or more adjacent contour-lines (Fig. 24, *f*). An over-hanging precipice is indicated by the lapping of the upper-level line over the lower-level line. Similarly any other characteristic feature of a country can be indicated.

There is no necessity for taking the level as the quantity which is constant over the intersecting surface. We might, for example, make the inclination of the tangent-plane to the vertical constant, and thus obtain another set of contour-curves by rolling this plane over the given surface.

As mentioned at the commencement of this paper, we can build up a solid, the surface of which represents the state of a substance with regard to three quantities. We may then lay down, upon this surface, contour-lines, each point of each of which indicates the relation between these quantities when a fourth quantity, characteristic of the line, remains constant.

Let us take, as a particular example, the thermodynamic surface representing the state of water-substance with regard to volume, pressure, temperature, entropy, and energy. If we choose any three of these quantities to be measured along the axes, the value of the remaining two at any point of the surface formed may be given by contour-lines. The model of the surface, with volume, entropy, and

energy, measured along the axes, has been constructed by Clerk-Maxwell, and is explained and figured in his *Theory of Heat.* Let us take volume, temperature, and pressure, as the quantities to be measured off. The surface so obtained was first studied, and some peculiarities connected with it pointed out, by Professor James Thomson. Suppose the surface to be cut by a plane of constant pressure, say p_1. We thus get a contour-line, the general nature of which is indicated in Fig. 25. At a low temperature the volume is small, the substance being solid. As the temperature rises the substance expands, until it reaches the liquifying point. Its volume then diminishes without rise of temperature until the substance is completely liquified. Its temperature then rises and its volume diminishes up to the maximum density point, after which it expands. When it reaches the boiling point its volume increases, but its temperature does not rise until the substance is entirely in the gaseous state, after which both increase together. The contour-lines for slightly less pressures, (p_2,p_3, in the Fig.), are approximately parallel to p_1, but lie entirely on the right-hand side of it, since for a given temperature the volume increases as the pressure diminishes and the freezing point is lowered and the boiling point is raised by pressure. The freezing point and boiling point approach as the pressure diminishes, until finally they coincide (see p_1, Fig. 26). After this (p_2, Fig. 26) the substance changes directly from the solid into the gaseous state. The line AB indicates the *triple-point* temperature, that is, the temperature at which portions of the substance in the three states, solid, liquid, and gaseous, can exist together in equilibrium. The length of the boiling-point line continually diminishes as the pressure is increased until, finally, there ceases to be a boiling-point (C, Figs. 25 and 27). The temperature at which this occurs is called the *critical temperature.* Similarly, we may assume a critical temperature for the solid-liquid condition. That is to say, there may be a temperature such that, if the temperature of the solid have a less value, no amount of pressure will lower the freezing point sufficiently to admit of liquifaction. It is, perhaps, too much to assume that there is a critical temperature for the solid-gaseous condition,—in other words, that at a certain pressure and temperature the whole mass of the solid will become gaseous without evaporation.

Now, suppose the surface to be cut by a plane of constant temperature. The contour-lines so obtained are ordinarily termed *isothermals.*

Let the temperature first be above the triple point but below the critical point. Then, the substance being taken in the gaseous state, as the pressure increases the volume diminishes until the boiling-point is reached. At this stage the volume decreases, without variation of pressure, until all the substance is liquified. After this, a very great increase of pressure is required to produce even a small decrease of volume. Such an isothermal is indicated by the line WXYZ, Fig. 27. If we take an isothermal below the triple-point, we find that the solid state is intermediate between the liquid and the gaseous. As the pressure increases the volume decreases until the point of sublimation is reached, when the pressure remains constant, the volume diminishing until all is solidified. Then the volume decreases slowly for increase of pressure until the melting-point is reached, when the pressure becomes constant, the volume diminishing, until all is melted, when the volume again decreases slowly for increase of pressure. Thus there are two kinds of isothermals having their transition stage at the triple-point temperature. We have seen that, similarly, at the triple-point pressure the transition stage for the two kinds of lines of equal pressure occurs. The form of the isothermals beyond the critical temperature is indicated in Fig. 27. Evidently, a second transition temperature for the isothermals is that of the solid-liquid critical temperature, if sublimation occurs at temperatures where liquifaction has ceased. It is probable that, as Professor James Thomson has indicated, the true form of the isothermals is not indicated by the part of the line parallel to the v-axis (Fig. 27), but by, for example, the waved line XY. Part of this line represents an unstable state since pressure and volume increase together. Hence the substance can only be obtained in nature in states represented by parts of this line.

On the surface, various contour-lines might be laid down. For example, we might have lines along which either of the quantities $\frac{dp}{dt}$ or $\frac{dp}{dv}$ was constant. Or we might have lines of constant energy, or of constant entropy. These latter are ordinarily termed adiabatics; that is, as the substance passes from one state to another along such a line, no heat enters or leaves it. The properties of these lines in a region where $\frac{dp}{dt}$ has a negative value are rather interesting. This condition is satisfied when the temperature of the substance is between the maximum-density point and the freezing-point.

This part of the surface is indicated in Fig. 28, which represents the projection of lines of equal volume upon the plane (p, t). MN, TC, TN, TS, are the projections respectively of the maximum-density curve, and the water-steam, water-ice, and ice-steam surfaces. In the region TMN, therefore, $\frac{dp}{dt}$ has a negative value. If the substance be in a state represented by a point in this part, and be allowed to expand adiabatically, its temperature rises until the maximum-density curve is reached. The adiabatic, however, cannot pass to the right-hand side of the curve, since the curve slopes upwards from left to right, and, in the region to the right, adiabatic expansion is accompanied by fall of temperature. Hence we find that two adiabatics may intersect on the surface in this region. That is, the substance may have the same temperature, volume, and pressure, in two different states corresponding to different amounts of intrinsic energy.

After the adiabatic reaches the maximum-density curve, the temperature may either rise or fall. Let us suppose that, (as indicated by Rücker), the intrinsic energy is such that, having done work while expanding, its temperature must fall. In this case it is evident (Fig. 29), that an isothermal can cut an adiabatic twice. Hence, we can have an isothermal steeper than an adiabatic at their point of intersection. In Fig. 29 MN is the maximum-density curve, the dotted curve is an isothermal, and the continuous curve is an adiabatic. Fig. 30 represents the contour-lines of equal pressure.

In the representation of physical properties by models, use might be made of tortuous curves. Thus, if we take two quantities, one of which is a parabolic function of the other, say x and y where $y^2 = ax$, and therefore $y\frac{dy}{dx} = \frac{a}{2}$, we may measure $\frac{dy}{dx}$ along a third rectangular axis, and so obtain a tortuous curve the projection of which on the plane (x, y) is a parabola, while its projection on the plane $\left(y, \frac{dy}{dx}\right)$ is a hyperbola. If y represent the time during which a body has been falling under gravity, and x represent the space described from rest, then the reciprocal of the third co-ordinate quantity gives the velocity acquired.

P.S.—From the experimental determination of the amount by which the maximum-density point is lowered by pressure, and the

theoretical determination of the steepness of the adiabatics in the plane (p, v), it seems that the latter are steeper than the former as regards inclination to the v-axis. Hence there is no point of maximum temperature on the adiabatic; but, on the other hand, there is a point of minimum temperature. This temperature for any given adiabatic, is that corresponding to the isothermal passing through the point of intersection of the adiabatic with the maximum-density curve. I have not altered the text above, however, as the remarks and figure may conceivably apply to some substance other than water.

The Theorems as far as Proposition 32, of the first book of Euclid's Elements, proved from First Principles.

By DAVID TRAILL, M.A., B.Sc.

Proposition 4.

Given * $AB = DE$, $AC = DF$, $\angle A = \angle D$.

Suppose you start from B, and walk along BA a certain distance a to A; then at A you turn at a certain angle into another road AC; then you walk along AC a certain distance b to C. Again you start from E, walk a distance a along ED; turn off at D into DF at the same angle as before; then walk the distance b along DF to F. Since you have gone through the same set of movements in the two cases, and since *the same cause always produces the same result*,† the results in the two cases must be the same, that is, you will arrive in both cases, at the same distance from the starting point. Hence BC = EF.

Proposition 5.

Given $AB = AC$.

From a certain point A, two lines AB, AC are drawn. Two points B, C equally distant from A are joined. The same causes which determine the size of $\angle$ B also determine the size of $\angle$ C. Hence $\angle$ B = $\angle$ C.

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* For figures see Mackay's *Elements of Euclid.*

† This axiom, as well as its converse, is assumed in every Science.

Proposition 6.

Given $\angle B = \angle C$.

From the two ends of a certain base line BC, two lines BA, CA are drawn making equal angles with BC. The two lines meet at A. The causes which determine the length of AB, also determine the length of AC. Hence AB = AC.

Proposition 8.

Given $AB = DE$, $AC = DF$, $BC = EF$.

Suppose you start from B, and walk along BA to A, then at A you turn at a certain angle A into AC, and walk along AC to C. Again you start from E, and walk along ED to D; then at D you turn at a certain angle D into DF, and walk along DF to F. Now BC = EF, that is, the results of the two sets of movements are equal. Hence the causes must be equal. In the one case the causes are BA, $\angle$ A, AC; in the other ED, $\angle$ D, DF. But BA = ED, and AC = DF. Hence $\angle$ A = $\angle$ D.

Propositions 13 and 14.

These two Propositions follow directly from the Definitions of a right angle and a straight line.

Proposition 15.

Given AB, CD two lines intersecting in E.

Since AB bisects the infinite plane, and CD bisects the same infinite plane; therefore the parts of the plane lying between these two lines are equal. Hence $\angle$ AEC = $\angle$ BED.

Propositions 16 and 17.

These two Propositions need not be considered as they are included in the 32nd.

Proposition 18.

Given AC greater than AB.

$\angle$ B cannot be equal to $\angle$ C; for if it were the effect would be that AB would be equal to AC. Hence $\angle$ B must either be greater or less than $\angle$ C. Now, suppose the smaller side AB gradually to diminish, $\angle$ B cannot, owing to this change in the length of AB, become equal to $\angle$ C, but must still remain greater or less. As AB continuously diminishes, angles B and C, if they change, must do so

continuously. Suppose AB at last to vanish, ∠ C also vanishes, but not ∠ B. Hence ∠ B must have been greater than ∠ C all along.

Proposition 19.

Given ∠ B greater than ∠ C.

AB cannot be equal to AC, therefore AB is either greater or less than AC. Now, suppose ∠ B to remain constant, and ∠ C to diminish gradually, AB will still remain either greater or less than AC; and when ∠ C vanishes AB also vanishes, but not AC. Hence AC must have been greater than AB all along.

Proposition 20.

To prove $BA + AC$ greater than BC.

This is axiomatic, therefore no proof is necessary.

Proposition 21.

I. *To prove $BA + AC$* greater than *$BD + DC$.*

This also is axiomatic.

The following proof, however, may be given :—

Join AD and produce it to meet BC in E.

Suppose BC to be an elastic cord with its ends fixed at B, C. Let it be displaced by the point of a rod which runs along EB. The result of this continuous displacement must be either continuous increase or continuous decrease, but the displacement to D has caused an increase in the length of BC, therefore the further displacement to A must cause further increase. Hence BA + AC is greater than BD + DC.

II. *To prove ∠ BDC greater than ∠ BAC.*

Let us now consider the effect of the continuous displacement of BC on the size of the contained angle. At E the contained angle = two right angles. At D the contained angle BDC is less than two right angles, therefore any further displacement means further decrease in the angle. Hence ∠ BDC is greater than ∠ BAC.

Proposition 24.

Given $AB = DE$, $AC = DF$, ∠ A greater than ∠ D.

BC is not equal to EF, therefore BC is either greater or less than EF. Now, suppose ∠ A to increase and ∠ D to diminish, BC will still remain greater or less than EF. If ∠ A become a straight angle, then BC = AB + AC, and if ∠ D vanish, then EF = the differ-

ence between ED and DF. Hence BC is now greater than EF, and must have been greater all along.

Proposition 25.

Given $AB = DE$, $AC = DF$, BC greater than EF.

When BC is the greatest possible, that is, when $BC = BA + AC$, then $\angle A = 2$ right angles; and when EF is the least possible, that is, when EF = the difference between ED and DF, then $\angle D$ vanishes. Hence when BC is greater than EF, then also is $\angle A$ greater than $\angle D$.

Proposition 26.

Given $\angle B = \angle E$, $\angle C = \angle F$, $BC = EF$.

In the one triangle any base line BC has been taken, and at its ends two angles B, C are formed by the two lines BA, CA, which meet at A; in the other triangle another base line EF, equal to BC, has been taken, and two angles, E, F respectively equal to B, C, are formed by the two lines ED, FD, which meet at D. We have the same causes in both cases, hence the results must be the same; that is, $BA = ED$, $AC = DF$.

Parallel Lines.

Definition:—Parallel lines are lines running in the same direction.

Axiom:—Through the same point two parallel lines cannot be drawn.

Lemma:—Parallel lines never meet. For if they did, then through the same point there could be drawn two parallel lines which is impossible.

Proposition 27.

Given $\angle AGH = \angle GHD$, or $\angle BGH = \angle GHC$.

First Proof. From the ends of the base line GH there are certain angles laid off on the one side, and also equal angles on the other; if the effect on the one side of GH is that the lines meet, then they must also meet on the other, which is impossible.

Second Proof. Suppose CD to revolve anti-clockwise round H through the angle DHG, then round G clockwise into the position AB. If, in its new position AB, it is not parallel to its original one, but meets it in a certain point, we should have a line after two revolutions equal and opposite, inclined at an angle to its original position, which is impossible.

Proposition 28.

I. *Given* $\angle EGB = \angle GHD$ *or* $\angle FHC = \angle HGA$.

If, because $\angle$ EGB $=$ $\angle$ GHD, AB, CD should meet towards BD, then because $\angle$ FHC $=$ $\angle$ HGA, AB, CD must meet towards A, C, which is impossible.

II. *Given* $\angle AGH + \angle GHC = 2$ *right angles, or* $\angle BGH + \angle GHD$ $= 2$ *right angles.*

First Proof. As in I.

Second Proof. Similar to Second Proof of Proposition 27.

Proposition 29.

Given AB parallel to CD.

I. $\angle$ AGH $=$ $\angle$ GHD, by a proof similar to Second Proof of Proposition 27.

II. $\angle$ BGH $+$ $\angle$ GHD $= 2$ right angles in the same way.

III. To prove $\angle$ EGB $=$ $\angle$ GHD.

First Proof. The size of $\angle$ EGB depends on two causes, the directions of the two lines GE, GB. The size of $\angle$ GHD depends on two causes, the directions of the two lines HG, HD; but the directions of GE, HG are the same, and also of GB, HD, because they are parallel. Hence $\angle$ EGB $=$ $\angle$ GHD.

Second Proof. There is nothing given as to the length of GH. Every proof, then, must be independent of the length of GH. Suppose GH gradually to diminish, and at last to vanish. Now when GH vanishes, that is when G, H coincide, then also AB, CD, since they are parallel, will coincide; for if they did not we should have two parallel lines drawn through the same point. Hence AB, CD coincide. Hence also $\angle$ EGB and $\angle$ GHD coincide and are equal.

Proposition 32.

Hamilton's Proof.* Suppose BC to revolve round B through $\angle$ B, till it coincides with BA, then round A through $\angle$ A till it reaches the position AC, then round C through $\angle$ C, till it returns to its original position, but with ends inverted. BC must have revolved through two right angles. Hence $\angle$ A $+$ $\angle$ B $+$ $\angle$ C $= 2$ right angles.

Professor Henrici in *Nature*, 13th March 1884, objects to this

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* See Casey's *Elements of Euclid*, pp. 244-246. The same proof, substantially, occurs in Playfair's edition of Simson's *Euclid*.

proof, on the ground that in the same way the angles of a spherical triangle might be proved equal to two right angles. On the contrary, a similar mode of proof will show that the angles of a spherical triangle are greater than two right angles. For we must now consider the revolution of planes containing the great circles of which the sides of the spherical triangle are arcs. Suppose, then, a plane by revolving to coincide in turn with the three sides of a spherical triangle. This plane in its three positions has always one point common, that is the centre of the sphere. The result of the three revolutions through the three spherical angles, is that the plane coincides with its original position, but with ends reversed. Now a plane can thus reverse its position by turning through two right angles, only on condition that it remains, during the revolution, perpendicular to the same fixed plane, that is that its axis of revolution is not subjected to tilting. Now, this is a condition that cannot be satisfied by a plane which coincides in turn with the three sides of a spherical triangle (except in the case when one side vanishes). Hence the three angles of a spherical triangle are greater than two right angles.

Ninth Meeting, July 11th, 1884.

Dr R. M. FERGUSON in the Chair.

Application of the Multiplication of Matrices to prove a Theorem in Spherical Geometry.

By Professor CHRYSTAL, University of Edinburgh.

The theorem in question is that if two of the diagonals of a spherical quadrilateral be quadrantal arcs, the third diagonal is also a quadrantal arc. (Fig. 31.)

Denote the direction cosines of the radius to the point 1 by l_1, m_1, n_1, &c., and $l_1 l_2 + m_1 m_2 + n_1 n_2$ by 12.

Then our conditions give $12 = 0$, $34 = 0$, and we have to prove $56 = 0$.

The equation to the plane 13 is

$\begin{vmatrix} x & y & z \\ l_1 & m_1 & n_1 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$, say $A_{13}x + B_{13}y + C_{13}z = 0$; and similarly the equation to the plane 24 is $A_{24}x + B_{24}y + C_{24}z = 0$.

Hence the direction cosines of 6 are the *full* minors of $\begin{vmatrix} A_{13} & B_{13} & C_{13} \\ A_{24} & B_{24} & C_{24} \end{vmatrix}$ divided by the square root of the sum of the squares of these minors. Hence the cosine 56 is $\begin{vmatrix} A_{13} & B_{13} & C_{13} \\ A_{24} & B_{24} & C_{24} \end{vmatrix} \times \begin{vmatrix} A_{14} & B_{14} & C_{14} \\ A_{23} & B_{23} & C_{23} \end{vmatrix}$ divided by the product of the square roots of the sums of the squares of the full minors of the two matrices.

Now by a double application of the multiplication of matrices

$$\begin{vmatrix} A_{13} & B_{13} & C_{13} \\ A_{24} & B_{24} & C_{24} \end{vmatrix} \times \begin{vmatrix} A_{14} & B_{14} & C_{14} \\ A_{23} & B_{23} & C_{23} \end{vmatrix}$$

$$= \begin{vmatrix} A_{13}A_{14} + B_{13}B_{14} + C_{13}C_{14} & A_{13}A_{23} + B_{13}B_{23} + C_{13}C_{23} \\ A_{24}A_{14} + B_{24}B_{14} + C_{24}C_{14} & A_{24}A_{23} + B_{24}B_{23} + C_{24}C_{23} \end{vmatrix}$$

$$= \begin{vmatrix} \begin{vmatrix} l_1 & m_1 & n_1 \\ l_3 & m_3 & n_3 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 & n_1 \\ l_4 & m_4 & n_4 \end{vmatrix} & \cdot & \begin{vmatrix} l_1 & m_1 & n_1 \\ l_3 & m_3 & n_3 \end{vmatrix} \times \begin{vmatrix} l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \\ \begin{vmatrix} l_2 & m_2 & n_2 \\ l_4 & m_4 & n_4 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 & n_1 \\ l_4 & m_4 & n_4 \end{vmatrix} & \cdot & \begin{vmatrix} l_2 & m_2 & n_2 \\ l_4 & m_4 & n_4 \end{vmatrix} \times \begin{vmatrix} l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \end{vmatrix}$$

$$= \begin{vmatrix} \begin{vmatrix} 11 & 14 \\ 31 & 34 \end{vmatrix} & \begin{vmatrix} 12 & 13 \\ 32 & 33 \end{vmatrix} \\ \begin{vmatrix} 21 & 24 \\ 41 & 44 \end{vmatrix} & \begin{vmatrix} 22 & 23 \\ 42 & 43 \end{vmatrix} \end{vmatrix}.$$

Since $12 = 0$, $34 = 0$, $11 = 22 = 33 = 44 = 1$, we have for the value of the last determinant

$$\begin{vmatrix} -13\cdot 14, & -13\cdot 23 \\ -24\cdot 14, & -24.23 \end{vmatrix} = 0;$$

which proves the proposition above stated.

The above process may be applied to the calculation of relations connecting the cosines of the arcs connected with the spherical quadrilateral in general. For example

$$57 = \frac{\begin{vmatrix} A_{14} & B_{14} & C_{14} \\ A_{23} & B_{23} & C_{23} \end{vmatrix} \times \begin{vmatrix} A_{12} & B_{12} & C_{12} \\ A_{34} & B_{34} & C_{34} \end{vmatrix}}{\sqrt{\begin{vmatrix} B_{14} & C_{14} \\ B_{23} & C_{23} \end{vmatrix}^2 + \&c. + \&c.} \times \sqrt{\begin{vmatrix} B_{12} & C_{12} \\ B_{34} & C_{34} \end{vmatrix} + \&c. + \&c.}}$$

$$= \frac{\begin{vmatrix} \begin{vmatrix} 11 & 12 \\ 41 & 42 \end{vmatrix} & \begin{vmatrix} 13 & 14 \\ 43 & 44 \end{vmatrix} \\ \begin{vmatrix} 21 & 22 \\ 31 & 32 \end{vmatrix} & \begin{vmatrix} 23 & 24 \\ 33 & 34 \end{vmatrix} \end{vmatrix}}{\sqrt{(A_{14}^2 + B_{14}^2 + C_{14}^2)(A_{23}^2 + B_{23}^2 + C_{23}^2) - (A_{14}A_{23} + B_{14}B_{23} + C_{14}C_{23})^2}\ \sqrt{\&c.}}.$$

Since $A_{14}^2 + B_{14}^2 + C_{14}^2 = (l_1^2 + m_1^2 + n_1^2)\ (l_4^2 + m_4^2 + n_4^2)$
$- (l_1l_4 + m_1m_4 + n_1n_4)^2 = 1 - 14^2,$

and $A_{14}A_{23} + B_{14}B_{23} + C_{14}C_{23} = \begin{vmatrix} 12 & 13 \\ 42 & 43 \end{vmatrix}$;

we get

$$57 = \frac{\begin{vmatrix} 1 & 34 & 12 \\ 14 & 13 & 24 \\ 23 & 24 & 13 \end{vmatrix}}{\sqrt{\left\{(1-14^2)(1-23^2) - \begin{vmatrix} 12 & 13 \\ 42 & 43 \end{vmatrix}^2\right\} \left\{(1-12^2)(1-34^2) - \begin{vmatrix} 13 & 14 \\ 23 & 24 \end{vmatrix}^2\right\}}};$$

In the particular case of the quadrantal quadrilateral this reduces to

$$57 = \frac{13^2 - 24^2}{\sqrt{\left\{(1-14^2)(1-23^2) - 13^2 24^2\right\} \left\{1 - (13 \cdot 24 - 14 \cdot 23)^2\right\}}};$$

from which 56 is obtained by interchanging 1 and 2.

On the Discrimination of Conics enveloped by the rays joining the corresponding points of two projective ranges.

By Professor CHRYSTAL.

It is evident in the first place as is pointed out by Steiner that the conic will be a parabola if, and cannot be a parabola unless the point at infinity on one range correspond to the point at infinity on the other, that is, the two ranges must be similar. This is the converse of the well-known proposition that a movable tangent to a parabola divides two fixed tangents similarly.

Steiner however does not take up the other cases, nor does Reye, or any other writer on the projective geometry of conics so far as I am aware.

We may however proceed in general as follows :

Join any two corresponding points P and P′. (Fig. 32.)

Project the range ω upon ω' by parallels to PP′.

We thus get a duplex range $\omega''\omega'$. Since this duplex range has already one double point P′P″, it must have another real double point, which can be easily constructed when k the power of the correspondence is given, if we observe that I projects into I″. To this second double point corresponds a tangent parallel to the tangent PP′.

From this it follows that the tangents to a curve of the second class occur in parallel pairs.

Remembering that points on the curve are the intersections of consecutive tangents we see that real points at infinity occur when the double points of the duplex range $\omega'\omega''$ are coincident.

Let $AI = i$, $B'J' = j'$, $B'I'' = x$; all with proper signs.

The condition for coincident double points is

$$I''J'^2 = -4k = -4\frac{B'I''}{AI}k;$$

$$\therefore \quad (j'-x)^2 = -4\frac{x}{i}k;$$

$$\therefore \quad x^2 - 2\left(j' - 2\frac{k}{i}\right)x + j'^2 = 0.$$

The roots of this equation must be real, that is, the condition for a hyperbola is

$$\left(j' - 2\frac{k}{i}\right)^2 - j'^2 > 0 \quad \text{or} \quad -k(k - ij') > 0$$

There are two distinct cases. If $k = -p^2$, then $p^2 + ij'$ must be >0. This will be satisfied if i and j' have the same sign, or if they have opposite signs and ij' be numerically $<p^2$. If $k = +p^2$, then must $p^2 - ij' > 0$; which will be satisfied if i and j' have opposite signs, or if they have the same sign provided ij' be numerically $<p^2$.

In a letter which I received not long ago from Professor Cremona, he gave me, in answer to an enquiry what construction he used for asymptotes to a conic generated by means of its tangents, the following construction, which is more elegant than the above, although it proceeds on much the same lines. Regarding my own, I may observe that it was meant to come at the very beginning of a course on the projective theory of conics, and was not supposed to assume any proposition regarding conics except the fundamental fact of their projective generation by the lines joining the corresponding points of two projective ranges.

Construzione degli assintoti della conica inviluppata dalle rette AA′, BB′, ... congiungenti i punti corrispondenti di due punteggiate projettive $r \equiv AB......, r' \equiv A'B'......$

Le coppie di tangenti parallele determinano sopra una tangente fissa r una involuzione di punti AA_1, BB_1,... il cui punto centrale R è il punto in cui r tocca la conica. Percio, se, in r, si prende $\overline{RP}^2 = \overline{RQ}^2 = RA.RA_1$, saranno P,Q i punti d'intersezione di r cogli assintoti.

La conica sia adunque data mediante due rette punteggiate projettive, sia S il punto ad esse commune, R il punto di contatto della prima, e T il punto della stessa prima punteggiata che corrisponde all' infinito della seconda. Allora prendendo nella prima retta $\overline{RP}^2 = \overline{RQ}^2 = RS.RT$, i punti P,Q appartengono agli assintoti.

On a Problem in Partition of Numbers.

By Professor CHRYSTAL.

At a recent meeting of the Royal Society of Edinburgh, Professor Tait proposed and solved the following problem:—

To calculate the number of Partitions of any number that can be made by taking any number from 2 up to another given number.

Let us denote by ${}_nP_r$ the number of partitions of r obtained by taking any of the numbers 2, 3, 4,......$(n-1)$, n. In the particular case $n=7$, $r=10$, the actual partitions are $3+7$, $4+6$, $5+5$; $2+2+6$, $2+3+5$, $2+4+4$, $3+3+4$; $2+2+2+4$, $2+2+3+3$; $2+2+2+2+2$; ten in all. Hence ${}_7P_{10}=10$.

The object proposed here is not to find an analytical expression for ${}_nP_r$, but to give a process for quickly calculating a table of double entry for it. The following has some advantages over the method given by Professor Tait although the result is in reality much the same.

$$\text{Since } \frac{1}{(1-x^2)(1-x^3)......(1-x^n)} = (1+x^2+x^{2.2}+x^{3.2}+......)$$
$$\times(1+x^3+x^{2.3}+x^{3.3}+......)$$
$$\times\,..............................$$
$$\times(1+x^n+x^{2n}+x^{3n}+......);$$

we have obviously

$$\frac{1}{(1-x^2)(1-x^3)\ldots\ldots(1-x^n)} = 1 + {}_n\mathrm{P}_1 x + {}_n\mathrm{P}_2 x^2 + \ldots\ldots + {}_n\mathrm{P}_r x^r + \ldots\ldots ;$$

$$\frac{1}{(1-x^2)(1-x^3)\ldots\ldots(1-x^n)(1-x^{n+1})} = 1 + {}_{n+1}\mathrm{P}_1 x + {}_{n+1}\mathrm{P}_2 x^2 + \ldots\ldots + {}_{n+1}\mathrm{P}_r x^r + \ldots\ldots ;$$

whence

$$(1-x^{n+1})(1 + {}_{n+1}\mathrm{P}_1 x + {}_{n+1}\mathrm{P}_2 x^2 + \ldots\ldots) = 1 + {}_n\mathrm{P}_1 x + {}_n\mathrm{P}_2 x^2 + \ldots\ldots .$$

Equating Coefficients we have

$$\begin{array}{ll} {}_{n+1}\mathrm{P}_1 = {}_n\mathrm{P}_1 & {}_{n+1}\mathrm{P}_{n+1} = {}_n\mathrm{P}_{n+1} + 1 \\ {}_{n+1}\mathrm{P}_2 = {}_n\mathrm{P}_2 & {}_{n+1}\mathrm{P}_{n+2} = {}_n\mathrm{P}_{n+2} + {}_n\mathrm{P}_1 \\ \ldots\ldots = \ldots\ldots & \ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots \\ {}_{n+1}\mathrm{P}_n = {}_n\mathrm{P}_n & {}_{n+1}\mathrm{P}_{n+r} = {}_n\mathrm{P}_{n+r} + {}_n\mathrm{P}_{n-r+1}. \end{array}$$

Remembering that

$${}_2\mathrm{P}_0 = 1, \quad {}_2\mathrm{P}_1 = 0, \quad {}_2\mathrm{P}_2 = 1, \quad {}_2\mathrm{P}_3 = 0, \text{ \&c.},$$

we can, therefore, tabulate (see fig. 32) the values of ${}_n\mathrm{P}_r$ on a piece of paper ruled into squares, as follows:—First, write in the upper line 1,0,1,0,1,0, &c. Through the second 1 draw the diagonal EF, then the numbers in the part of any column under this diagonal are simply the numbers *on the diagonal* repeated over and over again. These need not be written down. The lines to the right of the column are filled in thus—place a piece of paper cut in the form ABDC, with AB on the line GK, AC along a perpendicular to GK, and the blank in the line to be filled next to the last step of BD on that line. Then the blank is filled by adding the number above it to the number lowest in position on the immediate left of AC, whether that number lie in the first row 1,0,1,0,1, &c, or on the diagonal EF, or in the part of the line we are dealing with which has been already filled in. As ABDC is placed in the figure, the 25th square of the 20th line has just been filled in by adding 376 to 2.

La Tour d' Hanoï.

By R. E. Allardice, M.A., and A. Y. Fraser, M.A.

§ 1.—The following account of this problem is taken from the *Journal des Débats* for December 27th, 1883.

La poste nous a apporté ces jours-ci une petite boîte en carton peint, sur laquelle on lit: *la Tour d' Hanoï*, véritable casse-tête

annamite, rapporté du Tonkin par le professeur N. Claus (de Siam), mandarin du collège Li-Sou-Stian. Un vrai casse-tête en effet, mais intéressant. Nous ne saurions mieux remercier le mandarin de son aimable intention à l'égard d'un profane qu'en signalant *la Tour d' Hanoï* aux personnes patientes possédées par le démon du jeu.

On raconte que, dans le grand temple de Bénarès, au-dessous du dôme qui marque le centre du monde, on voit plantées dans une dalle d'airain trois aiguilles de diamant hautes d'une coudée et grosses comme le corps d'une abeille. Sur une de ces aiguilles, Dieu enfila au commencement des siècles 64 disques d'or pur, le plus large reposant sur l'airain, et les autres, de plus en plus étroits superposés jusqu'au sommet. C'est la tour de Brahma. Nuit et jour, les prêtres se succèdent, occupés à transporter la tour de la première aiguille de diamant sur la troisième sans s'écarter des règles fixes et immuables imposées par Brahma. Le prêtre ne peut déplacer qu'un seul disque à la fois; il ne peut poser ce disque que sur une aiguille libre ou au-dessus d'un disque plus grand. Lorsqu'en suivant strictement ces recommandations, les 64 disques auront été transportés de l'aiguille où Dieu les a placés sur la troisième, la tour et les brahmes tomberont en poussière et ce sera la fin du monde.

C'est évidemment cette légende qui a inspiré le mandarin du collège Li-Sou-Stian. La tour d'Hanoï, c'est la tour de Brahma; seulement les aiguilles de diamant sont remplacées par des clous et les disques d'or par des rondelles de bois. C'était plus prudent puisqu'il s'agit du Tonkin.

Les rondelles de taille décroissante sont au nombre de 8 seulement, et c'est bien assez. En opérant comme le font les brahmes, si la tour avait 64 étages, il faudrait tout simplement exécuter successivement un nombre de déplacemens exprimé par le nombre vertigineux de 18,446,744,073,709,551,615, ce qui exigerait plus de *cinq milliards de siècles !*

Avec 8 disques, il faut 255 déplacemens, ce qui, en attribuant une seconde à chaque déplacement, nécessite encore quatre minutes au moins pour transporter la tour.

En mettant en pratique la règle du jeu, on reconnaîtra vite que, pour déplacer 2 disques, il faut *trois* coups; pour 3 disques, *sept* coups, soit le double plus un; pour 4 disques, *quinze* coups, le double plus un, et ainsi de suite. Pour déplacer les 8, on voit qu'il faut *deux cent cinquante-cinq* coups.

Ce jeu ingénieux est fondé sur le problème élémentaire des combinaisons. Newton en a donné une formule générale très connue sous le nom de "Binôme de Newton." Mais les anciens, bien avant lui, avaient su trouver aussi l'expression correcte du nombre des combinaisons que l'on peut obtenir avec 11 lettres de l'alphabet. Le nombre des combinaisons possibles avec 4 lettres est égal à 2^4 diminué d'une unité; avec 5 lettres égale à 2^5 diminué d'une unité, etc. Avec 8 lettres, ou 8 disques, ce qui revient au même, 2^8 soit 256 diminué d'une unité, c'est-à-dire 255. Une tour de 9 disques nécessiterait de même le double des déplacemens plus un, ou, ce qui est la même chose, $2^9 - 1$, soit 513 déplacemens, etc.

La tour d'Hanoï nous a rappelé le jeu du baguenaudier, très étudié, entre autres jeux curieux, dans un ouvrage fort original que nous avons mentionné en son temps: les *Récréations mathématiques*, par M. Edouard Lucas, professeur au lycée Saint-Louis.

Ce souvenir m'est revenu fort à propos. Je tenais à découvrir le nom du mandarin, inventeur de la tour d'Hanoï. On n'est jamais trahi que par soi-même. Un mandarin, qui imagine un jeu fondé sur les combinaisons, doit sans cesse songer aux combinaisons, en voir et en mettre partout. Or, en permutant les lettres du signataire de la tour d'Hanoï, il me semble que l'on peut traduire, sans la moindre difficulté: professeur *N. Claus* (*de Siam*), mandarin du collège *Li-Sou-Stian:* Lucas d'Amiens, professeur du lycée Saint-Louis. Est-ce que moi aussi j'aurais trouvé mon problème? HENRI DE PARVILLE.

Taking an ordinary pile of eight brass weights and three sheets of paper, A,B,C, we may state the problem thus:—

The set of weights being on A, it is required to pile them in proper order on C by lifting only one at a time and laying it down either on an empty sheet or on a greater weight.

§ 2.—To find the number of moves required to shift n discs, two solutions are offered.

(1.) Let N_n be the number of moves required to shift n discs from A to C (or to B). To shift n discs to C the following plan must be followed: $n - 1$ discs are shifted to B (N_{n-1} moves), the n^{th} disc is now moved from A to C (one move), and then the $n - 1$ discs are shifted from B to C (N_{n-1} moves).

Hence $N_n = 2N_{n-1} + 1$;

$$
\begin{aligned}
\therefore\ N_n + 1 &= 2\left\{N_{n-1} + 1\right\} = 2\left\{2(N_{n-2} + 1)\right\} \\
&= \dots\dots\dots\dots\dots\dots \\
&= 2^{n-1}(N_1 + 1) = 2^{n-1}(1 + 1) \\
&= 2^n; \\
\therefore\ N_n \quad &= 2^n - 1.
\end{aligned}
$$

(2.) A little inspection will show that any disc has to be moved twice as often as the one immediately greater; and since the n^{th} disc is moved only once, it follows that

$$
\begin{aligned}
N_n &= 1 + 2 + 2^2 + \dots\dots + 2^{n-1} \\
&= \frac{2^n - 1}{2 - 1} = 2^n - 1.
\end{aligned}
$$

§ 3.—To accomplish the actual moves a simple rule may be given, as follows:—

To shift an even number of discs from A to C, move, by one step at a time, the odd numbers round ABC, counter-clockwise, the even numbers round ACB, clockwise.

To shift an odd number of discs the directions are reversed.

Note on Spherical Trigonometry.

By R. E. Allardice, M.A.

In the first volume of Gergonne's *Annales de Mathématiques* (1810-11), there is a paper by Lhuilier, in which he gives properties of the right-angled spherical triangle, analogous to the following properties of the right-angled plane triangle :

1. The square on the hypotenuse is equal to the sum of the squares on the other two sides;

2. If a perpendicular be drawn from the right angle to the hypotenuse, the square on each side is equal to the rectangle contained by the hypotenuse and the adjacent segment of the hypotenuse;

3. The squares on the sides are to one another as the adjacent segments of the hypotenuse;

4. The square on the perpendicular is equal to the rectangle contained by the segments of the hypotenuse;

5. The hypotenuse, the sides, and the perpendicular are in proportion.

Now the spherical triangle that has one angle equal to the sum of the other two, is in some respects analogous to the right-angled plane triangle; and it is one of the infinite number of spherical triangles which become right-angled when the radius of the sphere is made infinite. It is, for example, the triangle about which a semicircle may be described; and it is the triangle of maximum area, when two sides are given.

The following is a table containing the properties of the right-angled spherical triangle, given by Lhuilier, and the properties of the triangle referred to above, analogous in each case to the five properties of the right-angled plane triangle already given.

Denote the hypotenuse by c, the other sides by a and b, the perpendicular by h, and the segments of the base by l and m.

C a right angle.

1. $\text{Sin}^2 \tfrac{1}{2} c = \sin^2 \tfrac{1}{2} b \cos^2 \tfrac{1}{2} a + \sin^2 \tfrac{1}{2} a \cos^2 \tfrac{1}{2} b.$
2. $\text{Sin}^2 b : \sin c. \sin m = 1 : \cos l;$
   $\text{Sin}^2 a : \sin c. \sin l = 1 : \cos m.$
3. $\text{Sin}^2 a : \sin^2 b = \sin 2 l : \sin 2 m.$
4. $\text{Sin}^2 h : \sin l \sin m = 1 : \cos l \cos m.$
5. $\text{Sin}\, c : \sin a = \sin b : \sin h.$

$$C = A + B.$$

1. $\text{Sin}^2 \tfrac{1}{2} c = \sin^2 \tfrac{1}{2} a + \sin^2 \tfrac{1}{2} b.$
2. $\text{Tan}\, b \tan \tfrac{1}{2} b = \tan m \tan \tfrac{1}{2} c.$
   $\text{Tan}\, a \tan \tfrac{1}{2} a = \tan l \tan \tfrac{1}{2} c.$
3. $\text{Sin}^2 \tfrac{1}{2} a : \sin^2 \tfrac{1}{2} b = \sin l : \sin m.$
4. $\text{Tan}^2 \tfrac{1}{2} h = \tan \tfrac{1}{2} l \tan \tfrac{1}{2} m.$
5. $\text{Sin} \tfrac{1}{2} c : \sin \tfrac{1}{2} a = \sin \tfrac{1}{2} b : \tfrac{1}{2} \sin h.$

In both cases the above formulæ give the properties of the right-angled plane triangle, when the radius of the sphere is made infinite, the sine and the tangent of any angle then becoming equal to the angle, and the cosine of any angle unity. Formulæ may also be established for this triangle, analogous to those for the right-angled triangle, connecting any *three* of the elements of the triangle.

Fig. 1

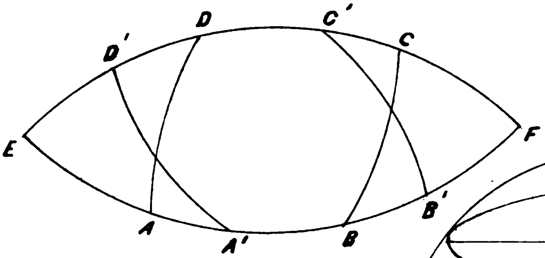


Fig. 2

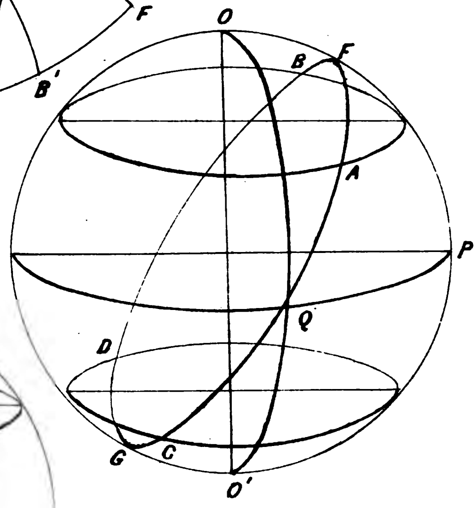


Fig. 3

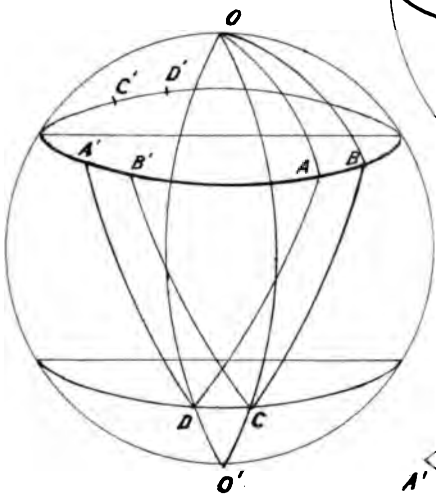


Fig. 4

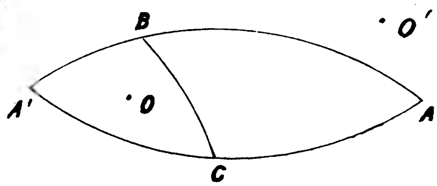


Fig. 5

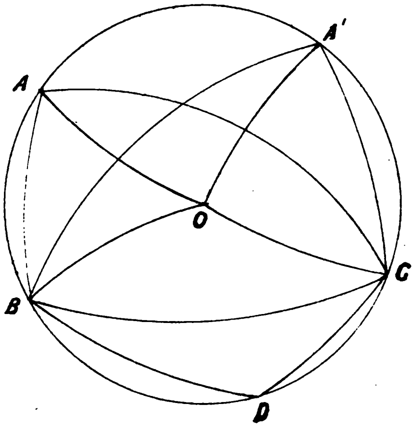


FIG. 6

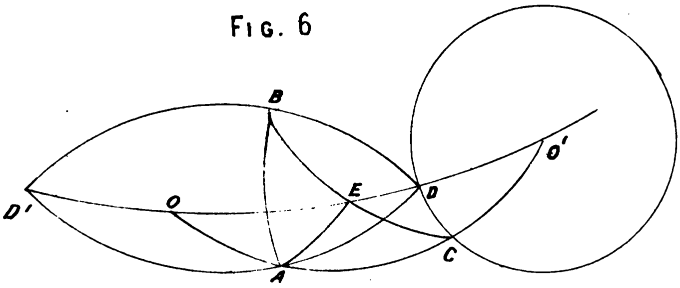


FIG. 7

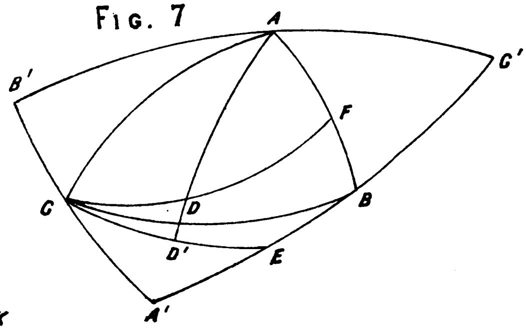


FIG. 8

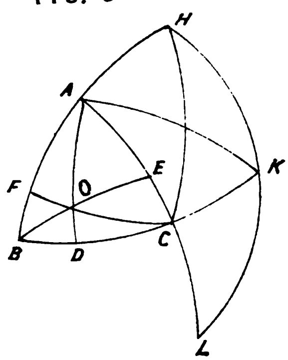


FIG. 9

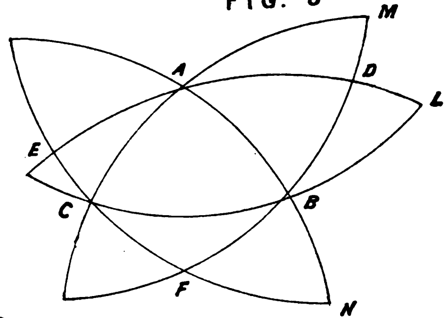
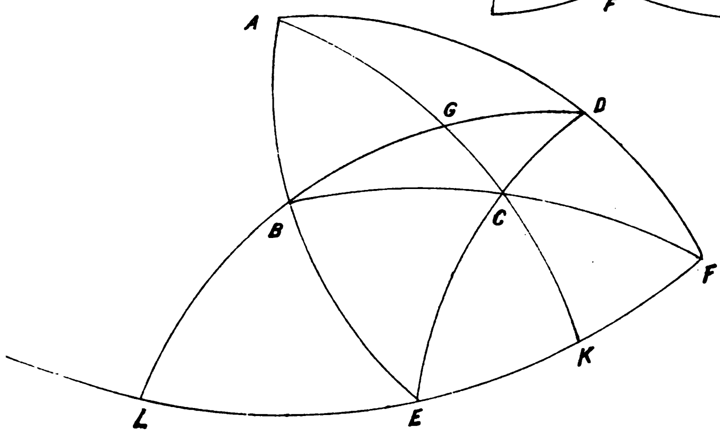


FIG. 10



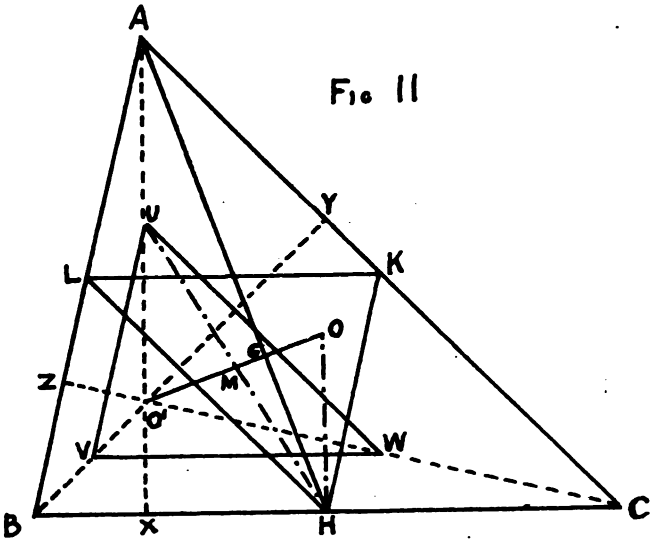


FIG 11

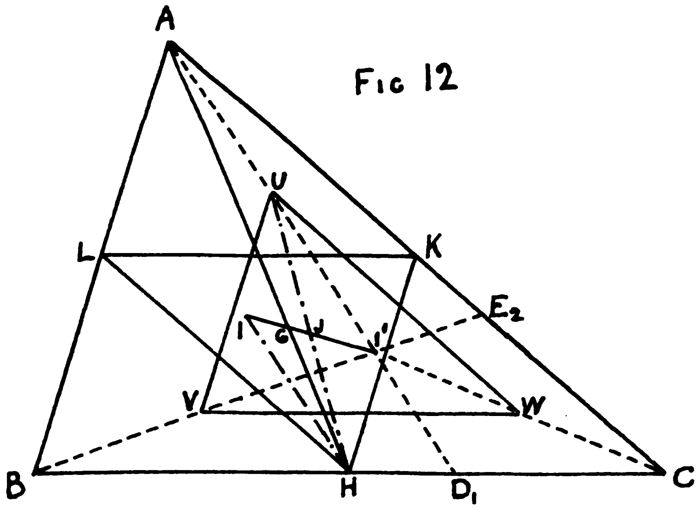


FIG 12

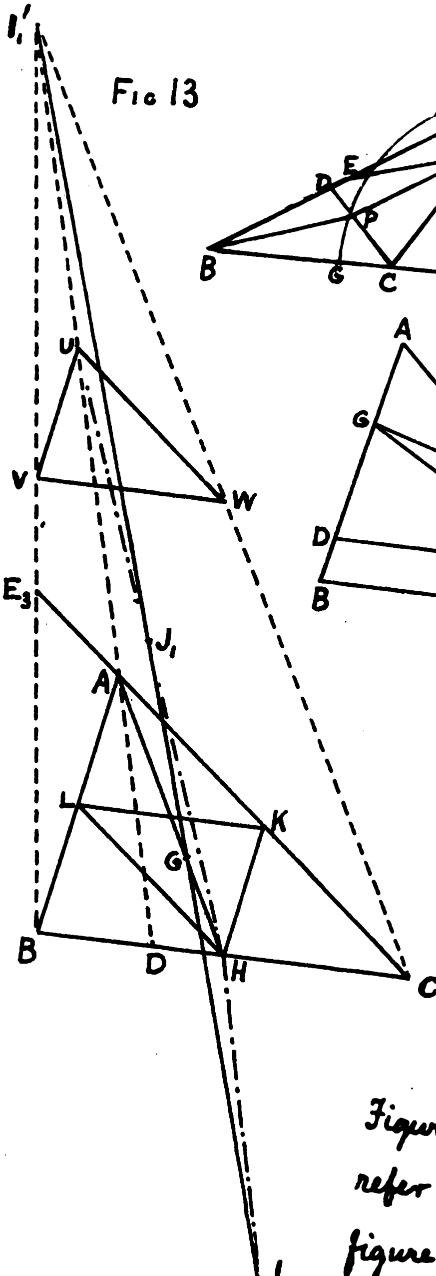


Fig 13

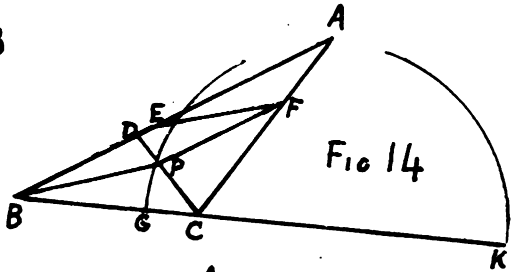


Fig 14

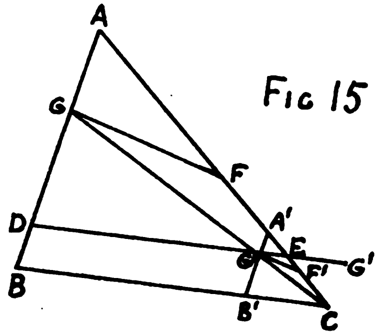


Fig 15

Figures 11, 12, 13 refer to p. 2; figure 14 to p. 5.

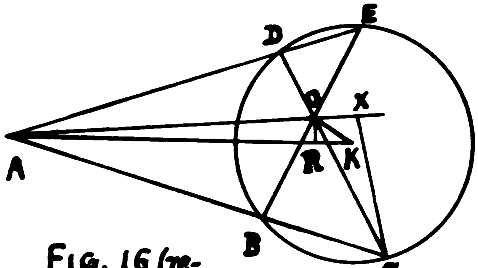


FIG. 16 (refers to p. 7.)

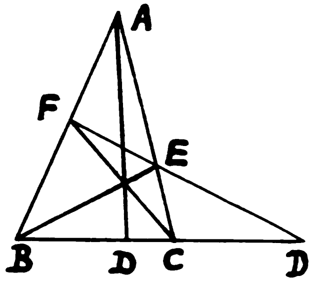


FIG. 17

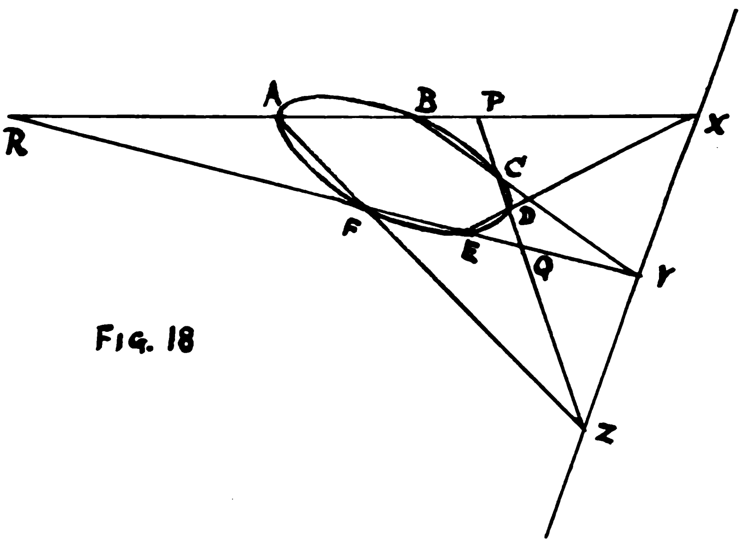


FIG. 18

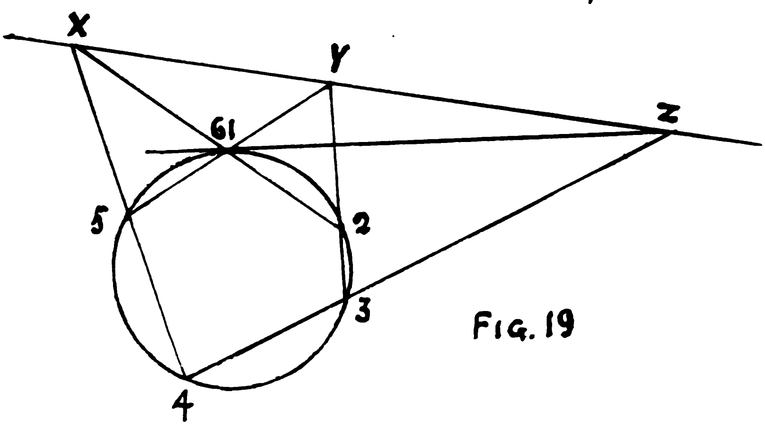


FIG. 19

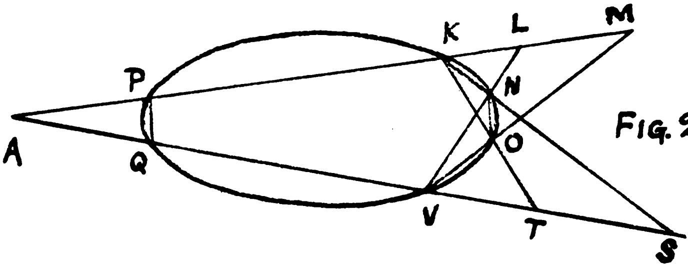


FIG. 20

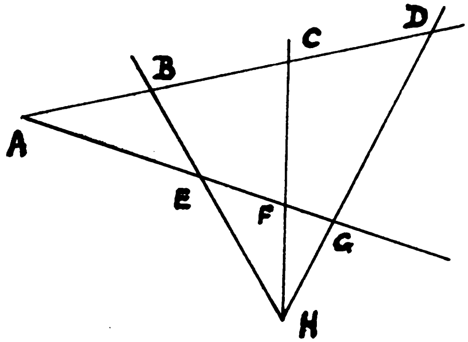


FIG. 21

FIG. 22

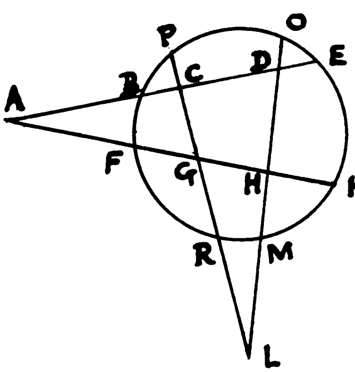


FIG. 23

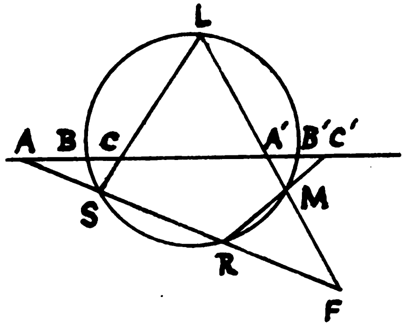


FIG. 24

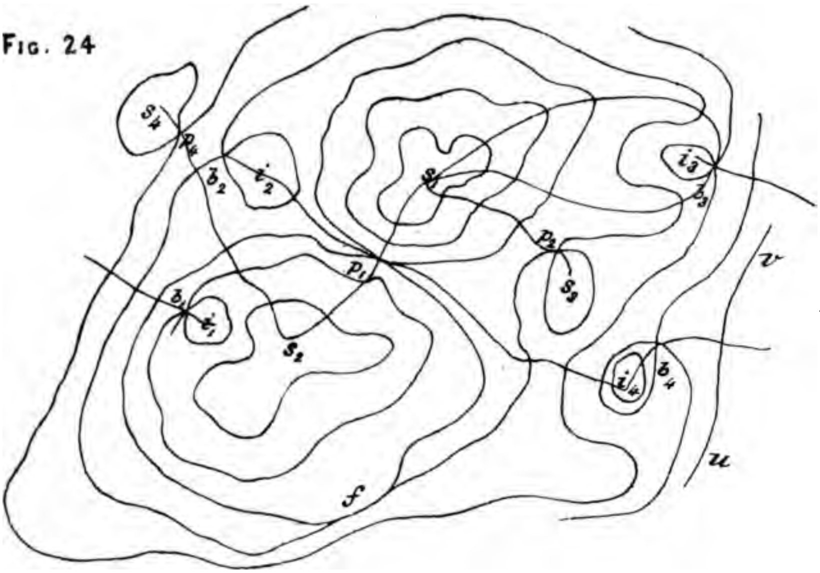


FIG. 25

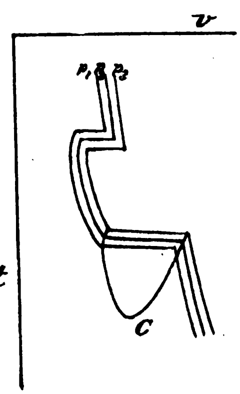


FIG. 26

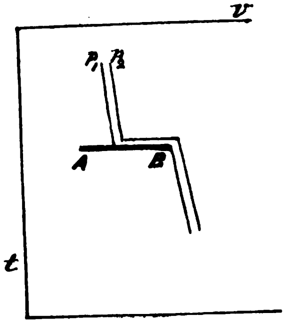


FIG. 27

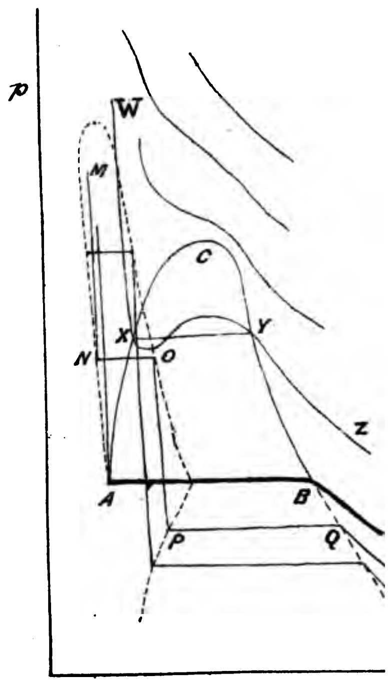


Fig. 28

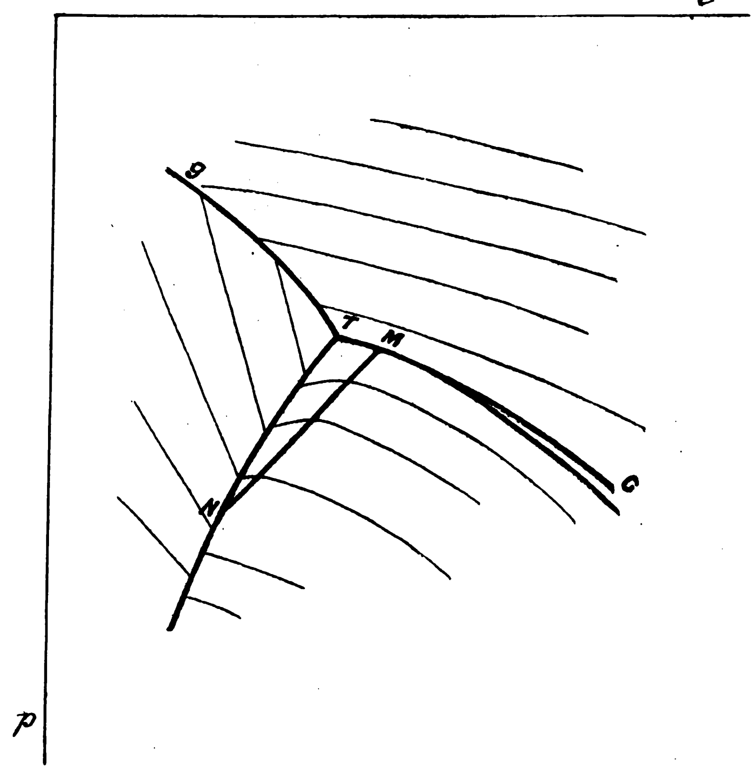


Fig. 29

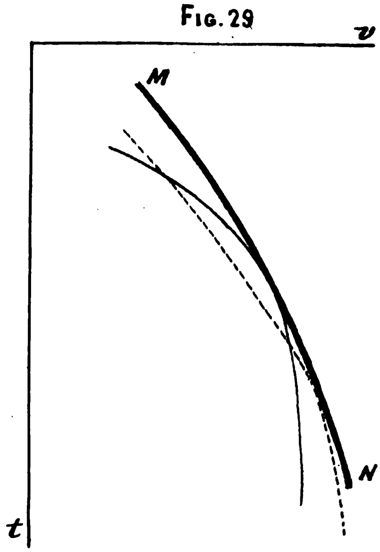


Fig. 30

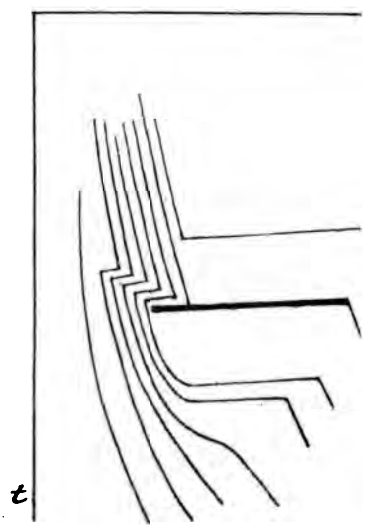


FIG. 31

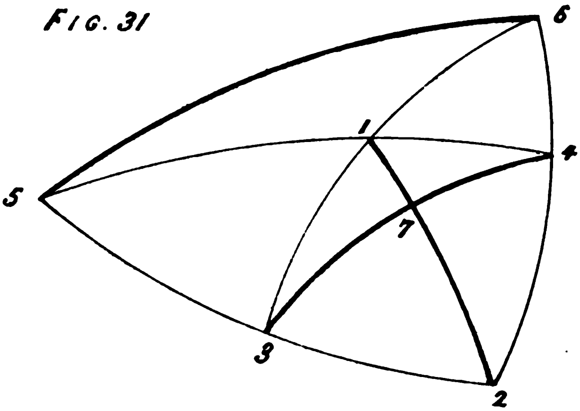
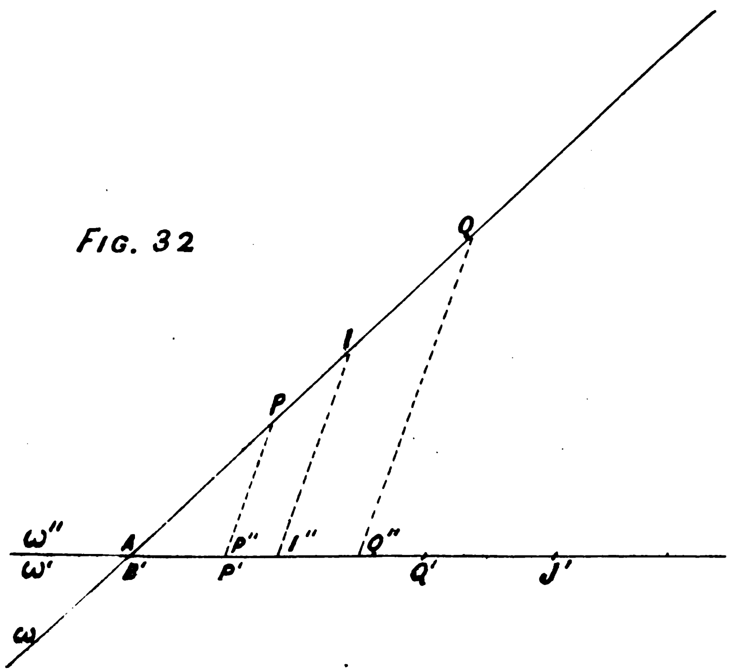


FIG. 32



VALUES OF T.

VALUES OF n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
2	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
3				1	1	1	2	1	2	2	2	2	3	2	3	3	3	3	4	3	4	4	4	4	5	4	5	5	5	5	6
4					2	1	3	2	4	3	5	4	7	5	8	7	10	8	12	10	14	12	16	14	19	16	21	19	24	21	27
5						2	3	3	5	5	7	7	10	10	13	14	17	18	22	23	28	29	34	36	42	44	50	53	60	63	71
6							4	3	6	6	9	9	14	13	19	20	26	27	36	36	47	49	60	63	78	80	97	102	120	126	149
7								4	6	7	10	11	16	17	23	26	33	37	47	52	64	72	86	96	115	127	149	166	192	212	245
8									7	7	11	12	18	19	27	30	40	44	58	64	82	91	113	126	155	171	207	230	274	303	358
9										8	11	13	19	21	29	34	44	51	66	75	95	110	134	155	189	215	258	296	349	398	468
10											12	13	20	22	31	36	48	55	73	83	107	123	154	177	220	251	306	351	422	481	575
11												14	20	23	32	38	50	59	77	90	115	135	168	197	243	283	344	401	481	558	665
12													21	23	33	39	52	61	81	94	122	143	180	211	264	306	377	440	533	619	746
13														24	33	40	53	63	83	98	126	150	188	223	278	327	401	473	573	672	809
14															34	40	54	64	85	100	130	154	195	231	290	341	422	497	607	712	863
15																41	54	65	86	102	132	158	199	238	298	353	436	518	631	746	904
16																	55	65	87	103	134	160	203	242	305	361	448	532	652	770	938
17																		66	87	104	135	162	205	246	309	368	456	544	666	791	962
18																			88	104	136	163	207	248	313	372	463	552	678	805	983
19																				105	136	164	208	250	315	376	467	559	686	817	997
20																					137	164	209	251	317	378					

G E A B K C D F

FIG. 33

STORAGE AREA